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On Jacobsthal and the Jacobsthal-Lucas sedenions and several identities involving these numbers

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Abstract

In this study, we define Jacobsthal and the Jacobsthal-Lucas sedenions and obtain a large variety of interesting identities for these numbers.

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1 Introduction

A great deal of attention is being paid to Jacobsthal and Jacobsthal-Lucas numbers because their interesting properties. Jacobsthal and Jacobsthal-Lucas numbers appear respectively as the integer sequences A001045 and A014551 from [8]. The classic Jacobsthal numbers in [5] are defined, for all nonnegative integers, by

$$J_n = J_{n-1} + 2J_{n-2}, \quad J_0 = 0, \quad J_1 = 1. \tag{1}$$

The classic Jacobsthal–Lucas numbers in [5] are defined, for all nonnegative integers, by

$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 2, \quad j_1 = 1. \tag{2}$$

The well-known division algebras arise from the quaternion and octonion algebras of dimension 4 and 8 (see [7]).
Szynal-Liana and Włoch [9] introduced the Jacobsthal quaternions and the Jacobsthal-Lucas quaternions and obtained some of their properties. Cerda-Morales [4] studied the third order Jacobsthal quaternions and the third order Jacobsthal-Lucas quaternions. Çimen and İpek [3] defined the Jacobsthal octonions and the Jacobsthal-Lucas octonions and presented some of their properties.
Sedenion algebra is a 16-dimensional CayleyDickson algebra and this algebra is presented in [6].
In this study, we define Jacobsthal and the Jacobsthal-Lucas sedenions and obtain a large variety of interesting properties for these numbers.

2 Main Results

Now, we define the nth Jacobsthal sedenion and Jacobsthal-Lucas sedenion numbers, respectively, by the following recurrence relations:

$$SJ_n = \sum_{s=0}^{15} J_{n+i} e_i, \tag{3}$$

and

$$Sj_n = \sum_{s=0}^{15} j_{n+i} e_i, \tag{4}$$

where J_n and j_n are the n th Jacobsthal number and Jacobsthal-Lucas number, respectively. By setting $i \equiv e_i$, where $i = 0, 1, \cdots, 15$, the following multiplication table is given (see [1] and [2]).

·	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	−0	3	−2	5	−4	−7	6	9	−8	−11	10	−13	12	15	−14
2	2	−3	−0	1	6	7	−4	−5	10	11	−8	−9	−14	−15	12	13
3	3	2	−1	−0	7	−6	5	−4	11	−10	9	−8	−15	14	−13	12
4	4	−5	−6	−7	−0	1	2	3	12	13	14	15	−8	−9	−10	−11
5	5	4	−7	6	−1	−0	−3	2	13	−12	15	−14	9	−8	11	−10
6	6	7	4	−5	−2	3	−0	−1	14	−15	−12	13	10	−11	−8	9
7	7	−6	5	4	−3	−2	1	−0	15	14	−13	−12	11	10	−9	−8
8	8	−9	−10	−11	−12	−13	−14	−15	−0	1	2	3	4	5	6	7
9	9	8	−11	10	−13	12	15	−14	−1	−0	−3	2	−5	4	7	−6
10	10	11	8	−9	−14	−15	12	13	−2	3	−0	−1	−6	−7	4	5
11	11	−10	9	8	−15	14	−13	12	−3	−2	1	−0	−7	6	−5	4
12	12	13	14	15	8	−9	−10	−11	−4	5	6	7	−0	−1	−2	−3
13	13	−12	15	−14	9	8	11	−10	−5	−4	7	−6	1	−0	3	−2
14	14	−15	−12	13	10	−11	8	9	−6	−7	−4	5	2	−3	−0	1
15	15	14	−13	−12	11	10	−9	8	−7	6	−5	−4	3	2	−1	−0

The conjugate of SJ_n and Sj_n are defined by

$$\overline{SJ_n} = J_n e_0 - J_{n+1} e_1 - J_{n+2} e_2 - J_{n+3} e_3 - ... - J_{n+15} e_{15}, \tag{5}$$

and

$$\overline{Sj_n} = j_n e_0 - j_{n+1} e_1 - j_{n+2} e_2 - j_{n+3} e_3 ... - j_{n+15} e_{15}. \tag{6}$$

The following identities are easy consequences from (3), (4), (5) and (6).

Theorem 2.1. *For $n \geq 1$, we have the following identities:*

1. $SJ_{n+1} = SJ_n + 2SJ_{n-1}$,
2. $SJ_n + \overline{SJ_n} = 2J_n e_0$,
3. $SJ_n^2 + SJ_n.\overline{SJ_n} = 2J_n.SJ_n$,
4. $SJ_n + Sj_n = 2SJ_{n+1}$,
5. $3SJ_n + Sj_n = 2^{n+1}(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15})$,
6. $Sj_{n+1} + 2Sj_{n-1} = 9Sj_n$.

Now, we will state the Binet's formulas for the Jacobsthal and Jacobsthal-Lucas sedenions. Noting that $J_n = \frac{1}{3}(2^n - (-1)^n)$, above (3) becomes

$$SJ_n = \frac{2^n}{3} A - \frac{(-1)^n}{3} B, \tag{7}$$

and then by using $j_n = 2^n + (-1)^n$, above (4) yields

$$Sj_n = 2^n A + (-1)^n B, \tag{8}$$

where $A = \sum_{s=0}^{15} 2^s e_s$ and $B = \sum_{s=0}^{15} (-1)^s e_s$. The formulas in (7) and (8) are called as Binet's forlmulas for the Jacobsthal and Jacobsthal-Lucas sedenions, respectively.

Theorem 2.2. *For $n \geq 1, r \geq 1$, we have the following identities:*

$$SJ_{n+1} + SJ_n = 2^n \left(e_0 + 2e_1 + 2^2 e_2 + 2^3 e_3 + ... + 2^{15} e_{15}\right), \tag{9}$$

$$\begin{aligned} SJ_{n+1} - SJ_n = & \frac{1}{3}\left[2^n \left(e_0 + 2e_1 + 2^2 e_2 + ... + 2^{15} e_{15}\right)\right. \\ & \left. + 2(-1)^n (e_0 - e_1 + e_2 - e_3 + ... - e_{15})\right], \end{aligned} \tag{10}$$

$$SJ_{n+r}+SJ_{n-r} = \frac{2^{n-r}\left(2^{2r}+1\right)}{3}\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right) + \frac{2\left(-1\right)^{n-r+1}}{3}\left(e_0-e_1+e_2-e_3+e_4-...-e_{15}\right), \quad (11)$$

$$SJ_{n+r}-SJ_{n-r}=\left(\frac{2^{n+r}-2^{n-r}}{3}\right)\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right). \quad (12)$$

Proof. If we consider (3) and (4), we have

$$SJ_{n+1}+SJ_n=\left(J_{n+1}+J_n\right)e_0+\left(J_{n+2}+J_{n+1}\right)e_1+...+\left(J_{n+16}+J_{n+15}\right)e_{15}.$$

With $j_{n+1}+j_n=3\left(J_{n+1}+J_n\right)=3.2^n$, we calculate the above sum as

$$SJ_{n+1}+SJ_n = 2^n\left(e_0+2e_1+...+2^{15}e_{15}\right).$$

If we again consider the definitions in equations (3)and (4), we get

$$SJ_{n+1}-SJ_n=\left(J_{n+1}-J_n\right)e_0+\left(J_{n+2}-J_{n+1}\right)e_1+...+\left(J_{n+16}-J_{n+15}\right)e_{15}.$$

Since $j_{n+1}-j_n=3\left(J_{n+1}-J_n\right)+4\left(-1\right)^{n+1}=2^n+2\left(-1\right)^{n+1}$, we can write this as

$$SJ_{n+1}-SJ_n=\frac{1}{3}\left[2^n\left(e_0+2e_1+...+2^{15}e_{15}\right)+2\left(-1\right)^n\left(e_0-e_1+e_2-e_3+...-e_{15}\right)\right].$$

Similarly, the identities (9) and (10) can be easily obtained by direct calculations.

Theorem 2.3. *For $n\geq 1, r\geq 1$, we have the following identities:*

$$Sj_{n+1}+Sj_n=3.2^n\left(e_0+2e_1+2^2e_2+2^3e_3+...+2^{15}e_{15}\right), \quad (13)$$

$$Sj_{n+1}-Sj_n = 2^n\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right) +2\left(-1\right)^{n+1}\left(e_0-e_1+e_2-e_3+...-e_{15}\right), \quad (14)$$

$$Sj_{n+r}+Sj_{n-r} = 2^{n-r}\left(2^{2r}+1\right)\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right) -2\left(-1\right)^{n-r}\left(e_0-e_1+e_2-...-e_{15}\right), \quad (15)$$

$$Sj_{n+r}-Sj_{n-r}=\left(2^{n+r}-2^{n-r}\right)\left(e_0+2e_1+2^2e_2+...+2^{15}e_{15}\right), \quad (16)$$

Proof. The proof of the identities $(13)-(16)$ of this theorem are similar to the proofs of the identities of Theorem 3, respectively, and are omitted here.

Theorem 2.4 (Cassini's identities)**.** *For Jacobsthal sedenions and Jacobsthal-Lucas sedenions the following identities are hold:*

$$SJ_{n+1}.SJ_{n-1} - SJ_n^2 = \frac{2^n (-1)^n}{3}\left[AB + \frac{BA}{2}\right], \tag{17}$$

$$SJ_{n-1}.SJ_{n+1} - SJ_n^2 = \frac{2^n (-1)^n}{3}\left[\frac{AB}{2} + BA\right], \tag{18}$$

$$Sj_{n+1}.Sj_{n-1} - Sj_n^2 = 2^{n-1} (-1)^{n+1} [6AB + 3BA], \tag{19}$$

and

$$Sj_{n-1}.Sj_{n+1} - Sj_n^2 = 2^{n-1} (-1)^{n+1} [3AB + 6BA], \tag{20}$$

where $A = \sum_{s=0}^{15} 2^s e_s$ *and* $B = \sum_{s=0}^{15}(-1)^s e_s$.

Proof. Using the Binet's formula in equation (17), we get

$$SJ_{n+1}.SJ_{n-1} - SJ_n^2 = \left(\frac{2^{n+1}}{3}A - \frac{(-1)^{n+1}}{3}B\right)\left(\frac{2^{n-1}}{3}A - \frac{(-1)^{n-1}}{3}B\right) - \left(\frac{2^n}{3}A - \frac{(-1)^n}{3}B\right)^2.$$

If necessary calculations are made, we obtain

$$SJ_{n+1}.SJ_{n-1} - SJ_n^2 = \frac{2^n (-1)^n}{3}\left[AB + \frac{BA}{2}\right].$$

In a similar way, using the Binet's formula in equation (18), we obtain

$$\begin{aligned} SJ_{n-1}.SJ_{n+1} - SJ_n^2 &= \left(\frac{2^{n-1}}{3}A - \frac{(-1)^{n-1}}{3}B\right)\left(\frac{2^{n+1}}{3}A - \frac{(-1)^{n+1}}{3}B\right) \\ &\quad - \left(\frac{2^n}{3}A - \frac{(-1)^n}{3}B\right)^2 \\ &= \frac{2^n (-1)^n}{3}\left[\frac{AB}{2} + BA\right] \end{aligned}$$

which is desired. Repeating same steps as in the proofs of (17) and (18), the proofs of (19) and (20) can be given.

Theorem 2.5 (Catalan's identities)**.** *For every nonnegative integer numbers* n *and* r *such that* $r \leq n$*, we get*

$$SJ_{n+r}.SJ_{n-r} - SJ_n^2 = \frac{2^n (-1)^n}{9}\left((-1)^r - 2^r\right)\left[AB(-1)^r - BA(2)^{-r}\right], \tag{21}$$

$$SJ_{n-r}.SJ_{n+r} - SJ_n^2 = \frac{2^n(-1)^n}{9}(2^r - (-1)^r)\left[AB(2)^{-r} - BA(-1)^{-r}\right], \quad (22)$$

$$Sj_{n+r}.Sj_{n-r} - Sj_n^2 = 2^n(-1)^n\left[AB(2^r(-1)^r - 1) + BA\left(2^{-r}(-1)^r - 1\right)\right], \quad (23)$$

and

$$Sj_{n-r}.Sj_{n+r} - Sj_n^2 = 2^n(-1)^n\left[AB\left(2^{-r}(-1)^r - 1\right) + BA\left(2^r(-1)^{-r} - 1\right)\right], \quad (24)$$

where $A = \sum_{s=0}^{15} 2^s e_s$ *and* $B = \sum_{s=0}^{15}(-1)^s e_s$.

Proof. Using the Binet's formula in equation (21), we get

$$\begin{aligned} SJ_{n+r}.SJ_{n-r} - SJ_n^2 &= \left(\frac{2^{n+r}}{3}A - \frac{(-1)^{n+r}}{3}B\right)\left(\frac{2^{n-r}}{3}A - \frac{(-1)^{n-r}}{3}B\right) \\ &\quad - \left(\frac{2^n}{3}A - \frac{(-1)^n}{3}B\right)^2 \\ &= \frac{2^n(-1)^n}{9}((-1)^r - 2^r)\left[AB(-1)^r - BA(2)^{-r}\right]. \end{aligned}$$

In a similar way, using the Binet's formula in equation (22), we obtain

$$\begin{aligned} SJ_{n-r}.SJ_{n+r} - SJ_n^2 &= \left(\frac{2^{n-r}}{3}A - \frac{(-1)^{n-r}}{3}B\right)\left(\frac{2^{n+r}}{3}A - \frac{(-1)^{n+r}}{3}B\right) \\ &\quad - \left(\frac{2^n}{3}A - \frac{(-1)^n}{3}B\right)^2 \\ &= \frac{2^n(-1)^n}{9}(2^r - (-1)^r)\left[AB(2)^{-r} - BA(-1)^{-r}\right]. \end{aligned}$$

The proofs of the identities (23) and (24) of this theorem are similar to the proofs of the identities (21) and (22) of theorem, respectively, and are omitted here.

Theorem 2.6 (d'Ocagne's identity)**.** *Suppose that* n *is a nonnegative integer number and* m *any natural number. If* $m > n$ *then:*

$$SJ_m.SJ_{n+1} - SJ_{m+1}SJ_n = \frac{1}{3}\left[2^m(-1)^n AB - 2^n(-1)^m BA\right] \quad (25)$$

and

$$Sj_m.Sj_{n+1} - Sj_{m+1}Sj_n = 3\left[-2^m(-1)^n AB + 2^n(-1)^m BA\right] \quad (26)$$

where $A = \sum_{s=0}^{15} 2^s e_s$ *and* $B = \sum_{s=0}^{15}(-1)^s e_s$.

Proof. Using the Binet's formula in equation (25), we have

$$\begin{aligned} SJ_m.SJ_{n+1} - SJ_{m+1}SJ_n &= \left(\frac{2^m}{3}A - \frac{(-1)^m}{3}B\right)\left(\frac{2^{n+1}}{3}A - \frac{(-1)^{n+1}}{3}B\right) \\ &\quad - \left(\frac{2^{m+1}}{3}A - \frac{(-1)^{m+1}}{3}B\right)\left(\frac{2^n}{3}A - \frac{(-1)^n}{3}B\right). \end{aligned}$$

If necessary calculations are made, we obtain

$$SJ_m.SJ_{n+1} - SJ_{m+1}SJ_n = \frac{1}{3}\left[2^m(-1)^n AB - 2^n(-1)^m BA\right].$$

In a similar way, using the Binet's formula in equation (26), we obtain

$$Sj_m.Sj_{n+1} - Sj_{m+1}Sj_n = 3\left[-2^m(-1)^n AB + 2^n(-1)^m BA\right].$$

Theorem 2.7. *For ordinary generating function of SJ_n defined by* (3)*, we have*

$$\mathcal{F}(x) = \frac{SJ_0 + (SJ_1 - SJ_0)x}{1 - x - 2x^2}. \tag{27}$$

Proof. Since generating function for Jacobsthal sedenions is

$$\mathcal{F}(x) = SJ_0x^0 + SJ_1x + SJ_2x^2 + ... + SJ_nx^n + ...$$

we see conclude that (27) by $\mathcal{F}(x) - x\mathcal{F}(x) - 2x^2\mathcal{F}(x)$.

Theorem 2.8. *The norms of nth Jacobsthal and Jacobsthal-Lucas sedenions are*

$$N(SJ_n) = \frac{1}{9}\left[43.692\left(32.767.0000\left(2^{2n}\right) + (2^n)(-1)^n\right) + 16\right] \tag{28}$$

and

$$N(Sj_n) = 43.692\left[32.767.0000\left(2^{2n}\right) - (2^n)(-1)^n\right] + 16 \tag{29}$$

respectively.

Proof. The norm of *nth* Jacobsthal sedenion is

$$N(SJ_n) = SJ_n\overline{SJ_n} = \overline{SJ_n}SJ_n = J_n^2 + J_{n+1}^2 + ... + J_{n+15}^2.$$

Making necessary calculations and using the equality $J_n = \frac{1}{3}(2^n - (-1)^n)$, we obtain (28) and (29).

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