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Generalized F-Shah-Rathie distribution and applications

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Abstract

In 1974, Shah and Rathie defined and studied a generalized-F density function (GFSR). This paper deals with this density and obtain its distribution function, mode, moments, reliability function $P(X < Y)$ when $X \sim$ Dagum and $Y \sim$ GFSR are independent, order statistics, Marshall-Olkin-Shah-Rathie distribution, and generalized gamma and beta-generated distributions. Maximum likelihood estimates are derived and applied to three real problems involving (a) Failure times of the air conditioning system, (b) Failure times of the Kevlar 49/epoxy strands with preasure of 90%, and (c) Bladder cancer patients data. The results show that the GFRS distribution is a good proposal for modeling these data sets.

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1 Introduction

In this section, we start by giving some definitions. The G-function is defined as

$$G_{p,q}^{m,n}\left[x\Big|_{b_1,\ldots,b_q}^{a_1,\ldots,a_p}\right]=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}x^s ds, \qquad (1)$$

where $x \neq 0$, an empty product is interpreted as unity, $0 \leq m \leq q$ and $0 \leq n \leq p$ (not both m and n zeros simultaneously). The parameters b_j, $j = 1, 2, \ldots, m$ and a_j, $j = 1, 2, \ldots, n$, are such that no pole of $\prod_{j=1}^{m}\Gamma(b_j - s)$ coincides with any pole of $\prod_{j=1}^{n}\Gamma(1 - a_j + s)$. See [9, pp. 143-144] for details about the contour L and conditions of convergence of the integral.

The H-function, which is a generalization of the G-function, is defined as

$$\begin{aligned} &H_{p,q}^{m,n}\left[x\Big|_{(b_1,B_1),\ldots,(b_m,B_m),(b_{m+1},B_{m+1}),\ldots,(b_q,B_q)}^{(a_1,A_1),\ldots,(a_n,A_n),(a_{n+1},A_{n+1}),\ldots,(a_p,A_p)}\right] \\ &= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-B_js)\prod_{j=1}^{n}\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=n+1}^{p}\Gamma(a_j-A_js)}x^s ds. \end{aligned} \qquad (2)$$

For more details about H-function, see [10].

We will also need the following integral (corrected form):

$$\begin{aligned} &\int_0^\infty x^{s-1} H_{p_1,q_1}^{m_1,n_1}\left[\eta x^{\lambda}\Big|_{{}_1(e_j,E_j)_{q_1}}^{{}_1(d_j,D_j)_{p_1}}\right] H_{p,q}^{m,n}\left[z x^{\lambda_1}\Big|_{{}_1(b_j,B_j)_q}^{{}_1(a_j,A_j)_p}\right] dx \\ &= \frac{\eta^{-\frac{s}{\lambda}}}{\lambda} H_{p+q_1,q+p_1}^{m+n_1,n+m_1}\left[z\eta^{-\frac{\lambda_1}{\lambda}}\Big|_{{}_1(b_j,B_j)_m,\ {}_1(1-d_j-\frac{s}{\lambda}D_j,\frac{\lambda_1}{\lambda}D_j)_{p_1},\ {}_{m+1}(b_j,B_j)_q}^{{}_1(a_j,A_j)_n,\ {}_1(1-e_j-\frac{s}{\lambda}E_j,\frac{\lambda_1}{\lambda}E_j)_{q_1},\ {}_{n+1}(a_j,A_j)_p}\right]. \end{aligned} \qquad (3)$$

For conditions of existence etc., see [10].

Definition 1.1. *A random variable X is called a generalized F-variate if its probability density function is given by (see [17])*

$$f(x,p,m,\alpha,h) = k\frac{x^{p-1}}{(1+\alpha x^h)^m}, \quad \alpha, m, p, h, x > 0, \qquad (4)$$

where

$$k = \frac{h\alpha^{\frac{p}{h}}}{B\left(\frac{p}{h}, m-\frac{p}{h}\right)}, \quad m > \frac{p}{h}, \qquad (5)$$

and $B(\cdot,\cdot)$ is the well-known beta function. We will call X having generalized-F-Shah-Rathie (GFSR) distribution.

The corresponding distribution function is given by (see Appendix)

$$F(x,p,m,\alpha,h) = \frac{k}{h\Gamma(m)\alpha^{\frac{p}{h}}} G_{2,2}^{1,2}\left[\alpha x^h \middle|\begin{matrix} \frac{p}{h}+1-m,\ 1 \\ \frac{p}{h},\ 0 \end{matrix}\right]. \tag{6}$$

The generalized F-distribution unifies some important sampling distributions. Particular cases of this distributions are described below:

(i) **Snedecor's F**. For $h=1$, $m=(m_1+m_2)/2$, $\alpha=m_1/m_2$, $p=m_1/2$ with m_1, m_2 positive integers, (4) and (6) reduce to:

$$f(x) = \frac{\left(\frac{m_1}{m_2}\right)^{\frac{m_1}{2}} x^{\frac{m_1}{2}-1}\left(1+\frac{m_1}{m_2}x\right)^{-\frac{m_1+m_2}{2}}}{\beta(\frac{m_1}{2},\frac{m_2}{2})},$$

$$F(x) = \frac{1}{\Gamma\left(\frac{m_1}{2}\right)\Gamma\left(\frac{m_2}{2}\right)} G_{2,2}^{1,2}\left[\frac{m_1}{m_2}x \middle|\begin{matrix} \frac{m_1}{2}+1-\frac{m_1+m_2}{2},1 \\ \frac{m_1}{2},0 \end{matrix}\right].$$

(ii) **Beta second kind**. For $h=1$, $\alpha=1$, $p>0$, $m>p$, (4) and (6) yield:

$$f(x) = \frac{1}{\beta(p,m-p)} x^{p-1}(1+x)^{-m},$$

$$F(x) = \frac{1}{\Gamma(p)\Gamma(m-p)} G_{2,2}^{1,2}\left[x \middle|\begin{matrix} p+1-m,1 \\ p,0 \end{matrix}\right].$$

(iii) **Folded Student-t**. For $h=2$, $\alpha=1/N$, $p=1$, $m=(N+1)/2$, (4) and (6) become:

$$f(x) = \frac{2}{\sqrt{N}\beta\left(\frac{1}{2},\frac{N}{2}\right)}\left(1+\frac{x^2}{N}\right)^{-\left(\frac{N+1}{2}\right)},$$

$$F(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{N}{2}\right)} G_{2,2}^{1,2}\left[\frac{1}{N}x^2 \middle|\begin{matrix} \frac{N}{2},1 \\ \frac{1}{2},0 \end{matrix}\right].$$

(iv) **Folded Cauchy**. For $h=2$, $p=1$, $m=1$, $\alpha=1$, (4) and (6) become:

$$f(x) = \frac{2}{\beta\left(\frac{1}{2},\frac{1}{2}\right)}(1+x^2)^{-1} = \frac{2}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}(1+x^2)^{-1} = \frac{2}{\pi(1+x^2)},$$

$$F(x) = \frac{1}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)} G_{2,2}^{1,2}\left[x^2 \middle|\begin{matrix} \frac{1}{2},1 \\ \frac{1}{2},0 \end{matrix}\right] = \frac{1}{\pi} G_{2,2}^{1,2}\left[x^2 \middle|\begin{matrix} \frac{1}{2},1 \\ \frac{1}{2},0 \end{matrix}\right].$$

(v) **Dagum** ([5]). For $p = v_1 v_3$, $\alpha = 1/v_2$, $h = v_3$, $m = v_1 + 1$, (4) and (6) give:

$$f(x) = \frac{v_1 v_2 v_3 x^{-v_3-1}}{(1 + v_2 x^{-v_3})^{v_1+1}} = \frac{v_1 v_2 v_3 x^{-v_3-1}}{v_2^{v_1+1} x^{-v_3(v_1+1)} \left(1 + \frac{1}{v_2} x^{v_3}\right)^{v_1+1}}$$

$$= \frac{v_1 v_3 x^{v_1 v_3 - 1}}{v_2^{v_1} \left(1 + \frac{x^{v_3}}{v_2}\right)^{v_1+1}}, \tag{7}$$

$$F(x) = (1 + v_2 x^{-v_3})^{-v_1} = \frac{1}{(v_2 x^{-v_3})^{v_1} \left(1 + \frac{x^{v_3}}{v_2}\right)^{v_1}}$$

$$= \frac{x^{v_1 v_3}}{v_2^{v_1} \left(1 + \frac{x^{v_3}}{v_2}\right)^{v_1}} = \frac{1}{\Gamma(v_1)} G^{1,1}_{1,1}\left[\frac{x^{v_3}}{v_2}\middle|\begin{matrix}1\\ v_1\end{matrix}\right], \tag{8}$$

by using [9, p. 149]

$$G^{1,2}_{2,2}\left[z\middle|\begin{matrix}0,\ 1\\ a,\ 0\end{matrix}\right] = G^{1,1}_{1,1}\left[z\middle|\begin{matrix}1\\ a\end{matrix}\right] = z^a (1+z)^{-a}\Gamma(a).$$

(vi) **Burr III**. Taking $v_2 = 1$ in (7) and (8) and changing v, by α and v_3 by σ, we have:

$$f(x) = \frac{\alpha\sigma x^{-\sigma-1}}{(1 + x^{-\sigma})^{\alpha+1}} = \frac{\alpha\sigma x^{-\sigma-1}}{x^{-\sigma(\alpha+1)}(1 + x^{\sigma})^{\alpha+1}} = \frac{\alpha\sigma x^{\alpha\sigma-1}}{(1 + x^{\sigma})^{\alpha+1}},$$

$$F(x) = \frac{x^{\alpha\sigma}}{(1 + x^{\sigma})^{\alpha}}.$$

(vii) **Burr XII-Singh-Maddala** ([18]). For $p = h$, on using [13, p. 390 (125)], (4) and (6) give:

$$f(x) = \frac{p\alpha(m-1)x^{p-1}}{(1 + \alpha x^p)^m}, \tag{9}$$

$$F(x) = 1 - (1 + \alpha x^p)^{1-m}, \quad m > 1. \tag{10}$$

(viii) **Pareto**. For $p = 1$, $\alpha = 1/c$ and $m = a + 1$, (9) and (10) give:

$$f(x) = \frac{ac^a}{(x + c)^{a+1}}, \quad x, a, c > 0, \tag{11}$$

$$F(x) = 1 - \frac{c^a}{(x + c)^a}. \tag{12}$$

(ix) For $p=1$, $\alpha=t_*^{-1/q}$, $h=1/q$ and $m=q+1$, (4) and (6) yield

$$f(x)=\frac{1}{t_*}\left[1+\left(\frac{x}{t_*}\right)^{\frac{1}{q}}\right]^{-q-1}, \tag{13}$$

$$F(x)=\left[1+\left(\frac{t_*}{x}\right)^{\frac{1}{q}}\right]^{-q}. \tag{14}$$

The density and distribution functions given in (13) and (14) were studied by [19] and applied to data concerning sediment deposits in a water flow reservoir.

The paper is divided as follows: Section 2 deals with the determination of mode and a few graphs of the GFSR-distribution for various values of the parameters. In Section 3, moments about the origin are given while Section 4 deals with the reliability $P(X<Y)$ when $X\sim$ Dagum and $Y\sim$ GFSR are independent. Section 5, 6, 7 and 8 deal respectively with order statistics, Marshall-Olkin-Shah-Rathie distribution, generalized gamma- and beta-generated GFSR-distributions. In Section 9, the maximum likelihood estimation method is used to estimate the parameters and applied to analize real data involving three problems. In the last Section 10, we conclude the paper.

2 Mode and graphs

From (4), we have

$$\begin{aligned}\frac{\partial f}{\partial x}&=k[-x^{p-1}m(1+\alpha x^h)^{-m-1}\alpha h x^{h-1}+(1+\alpha x^h)^{-m}(p-1)x^{p-2}]\\&=kx^{p-2}(1+\alpha x^h)^{-m-1}[-m\alpha h x^h+(1+\alpha x^h)(p-1)].\end{aligned} \tag{15}$$

For maximum or minimum,

$$(p-1)+\alpha x^h(p-1-mh)=0$$

giving

$$x=\left[\frac{p-1}{\alpha(mh-p+1)}\right]^{\frac{1}{h}}=A,\ \text{suppose}. \tag{16}$$

For $x>0$, there are two cases: (1) $(p-1)>0$ and $(mh-p+1)>0$, resulting $mh>(p-1)>0$; (2) $(p-1)<0$ and $(mh-p+1)<0$, resulting $mh<(p-1)<0$, which is not possible.

Differentiating (15), we have

$$\frac{\partial^2 f}{\partial x^2} = kx^{p-1}(1+\alpha x^h)^{-m-1}[-m\alpha h^2 x^{h-1} + (p-1)\alpha h x^{h-1}] \\ + [-m\alpha h x^h + (1+\alpha x^h)(p-1)] \\ \times k[(p-2)x^{p-3}(1+\alpha x^h)^{-m-1} - x^{p+h-3}\alpha h(m+1)(1+\alpha x^h)^{m-2}] \quad (17)$$

giving

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{x=A} = hA^{p-2}(1+\alpha A^h)^{-m-1}A^{h-1}\alpha h[-mh+p-1] < 0. \quad (18)$$

Hence, the mode is given by (16) for $p > 1$.

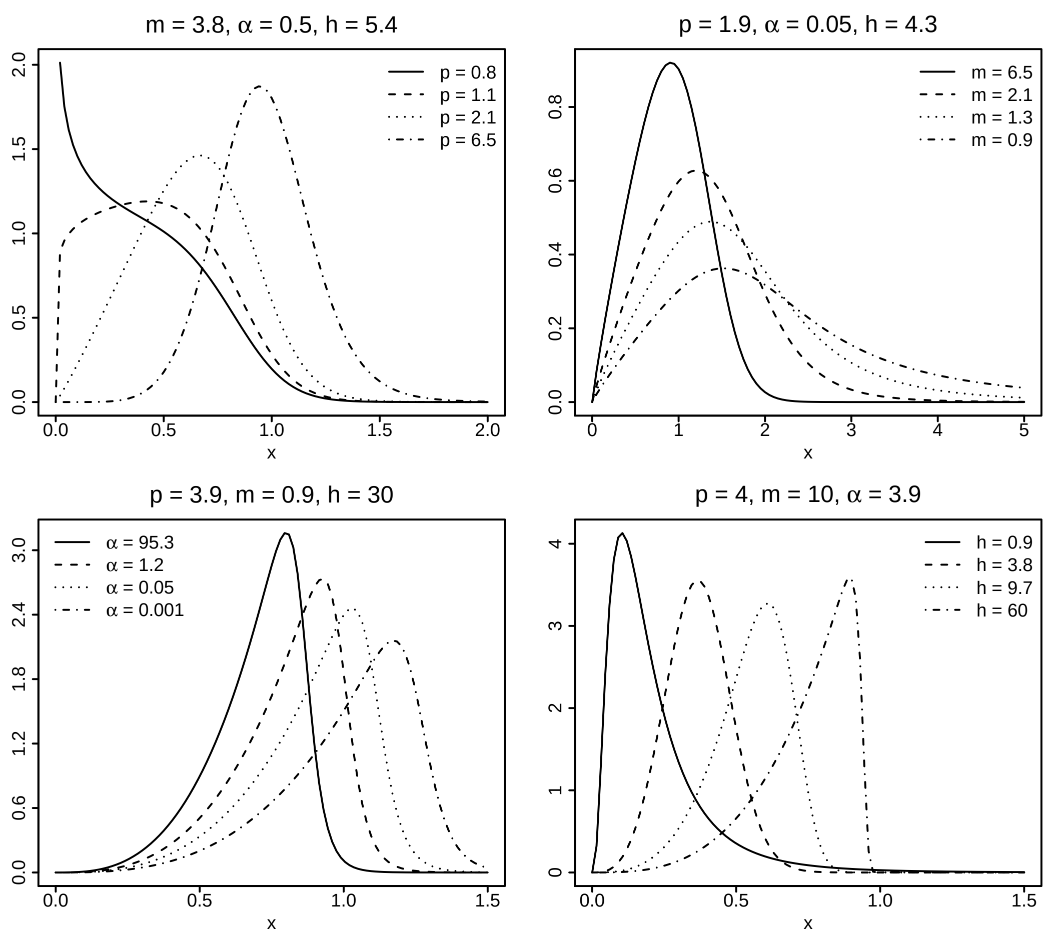

Figure 1: Some shapes for the GFSR density.

3 Moments

The $(s-1)$-th moments of X are given by

$$E(X^{s-1}) = \frac{\Gamma\left(\frac{s+p-1}{h}\right)\Gamma\left(m-\frac{p-1+s}{h}\right)}{\alpha^{\frac{(s-1)}{h}}\Gamma\left(\frac{p}{h}\right)\Gamma\left(m-\frac{p}{h}\right)}, \tag{19}$$

for $\mathrm{Re}(s) < mh - p + 1$, $m > p/h$.

Proof. We have (see also [17])

$$\begin{aligned} E(X^{s-1}) &= k\int_0^\infty \frac{x^{s+p-2}}{(1+\alpha x^h)^m}dx \\ &= \frac{h\alpha^{\frac{p}{h}}}{\beta\left(\frac{p}{h}, m-\frac{p}{h}\right)} \frac{\beta\left(\frac{p+s-1}{h}, m-\frac{p+s-1}{h}\right)}{h\alpha^{\frac{(p+s-1)}{h}}} \\ &= \frac{\Gamma\left(\frac{p+s-1}{h}\right)\Gamma\left(m-\frac{p+s-1}{h}\right)}{\alpha^{\frac{(s-1)}{h}}\Gamma\left(\frac{p}{h}\right)\Gamma\left(m-\frac{p}{h}\right)}, \end{aligned} \tag{20}$$

for $m > p/h$ and $1-p < \mathrm{Re}(s) < mh+1-p$. □

4 Reliability

Let $X \sim \mathrm{Dagum}(v_1, v_2, v_3)$ and $Y \sim \mathrm{GFSR}(p, m, \alpha, h)$ be independent, then

$$P(X < Y) = \int_o^\infty F_X(t) f_Y(t)dt = \int_0^\infty \frac{t^{v_1 v_3}}{v_2^{v_1}\left(1+\frac{t^{v_3}}{v_2}\right)^{v_1}} \frac{kt^{p-1}}{(1+\alpha t^h)^m}dt, \tag{21}$$

where $k = \dfrac{h\alpha^{\frac{p}{h}}}{\beta\left(\frac{p}{h}, m-\frac{p}{h}\right)}$ and $m > p/h$.

Using

$$(1+z)^{-a} = \frac{1}{\Gamma(a)} H_{11}^{11}\left[z\big|_{(0,1)}^{(1-a,1)}\right] = \frac{1}{\Gamma(a)} G_{11}^{11}\left[z\big|_{0}^{1-a}\right], \tag{22}$$

we have

$$P(X<Y)=\frac{k}{v_2^{v_1}}\frac{1}{\Gamma(v_1)\Gamma(m)}\int_0^\infty t^{v_1v_3+p-1} H_{1,1}^{1,1}\left[\frac{t^{v_3}}{v_2}\middle|\begin{matrix}(1-v_1,1)\\(0,1)\end{matrix}\right] H_{1,1}^{1,1}\left[\alpha t^h\middle|\begin{matrix}(1-m,1)\\(0,1)\end{matrix}\right]dt. \tag{23}$$

Taking $m=n=p=g=m_1=n_1=p_1=g_1=1$, $s=v_1v_3+p$, $z=1/v_2$, $\lambda_1=v_3$, $\eta=\alpha$, $\lambda=h$, $A_1=B_1=D_1=E_1=1$, $a_1=1-v_1$, $b_1=0$, $d_1=1-m$, $e_1=0$ in (3) to evaluate the integral, we get

$$P(X<Y)=\frac{k}{v_2^{v_1}\Gamma(v_1)\Gamma(m)h\alpha^{\frac{v_1v_3+p}{h}}}H_{2,2}^{2,2}\left[\frac{1}{v_2\alpha^{\frac{v_3}{h}}}\middle|\begin{matrix}(1-v_1,1),(1-\frac{v_1v_3+p}{h},\frac{v_3}{h})\\(0,1),(m-\frac{v_1v_3+p}{h},\frac{v_3}{h})\end{matrix}\right]. \tag{24}$$

5 Order Statistics

The n-th and 1-st order statistics are given by

$$\begin{aligned}F_n(x)&=F^n(x),\\ f_n(x)&=nF^{n-1}(x)f(x),\\ F_1(x)&=1-[1-F(x)]^n,\\ f_1(x)&=n[1-F(x)]^{n-1}f(x),\end{aligned}$$

where $f(x)$ and $F(x)$ are given in (1) and (3).

6 Marshall-Olkin-Shah-Rathie Distributions

Using (6) in

$$G(x)=\frac{F(x)}{F(x)+\beta\overline{F}(x)}, \tag{25}$$

we have the MOSR distribution as

$$G(x)=\frac{kG_{2,2}^{1,2}\left[\alpha x^h\middle|\begin{matrix}\frac{p}{h}+1-m,1\\\frac{p}{h},0\end{matrix}\right]}{h\Gamma(m)\alpha^{\frac{p}{h}}\beta+(1-\beta)kG_{2,2}^{1,2}\left[\alpha x^h\middle|\begin{matrix}\frac{p}{h}+1-m,1\\\frac{p}{h},0\end{matrix}\right]}, \tag{26}$$

where $k=\dfrac{h\alpha^{\frac{p}{h}}}{\beta\left(\frac{p}{h},m-\frac{p}{h}\right)}$.

As particular cases, we have the following distributions:

(a) Marshall-Olkin-Dagum distribution: Taking $p = v_1 v_3$, $\alpha = 1/v_2$, $h = v_3$ and $m = v_1 + 1$ in (26), we have

$$G(x) = \frac{1}{\beta\left(1 + v_2 x^{-v_3}\right)^{v_1} + (1-\beta)}. \tag{27}$$

(b) Marshall-Olkin-Burr III distribution: Taking $v_2 = 1$ in (27), we get

$$G(x) = \frac{1}{\beta(1 + x^{-v_3})^{v_1} + (1-\beta)}. \tag{28}$$

7 Generalized gamma-generated GFSR-distributions

Using the following distribution function generalized-gamma representation

$$H_1(x) = \frac{cb^{a/c}}{\Gamma\left(\frac{a}{c}\right)} \int_0^{-ln(1-F(x))} w^{a-1} e^{-bw^c} dw \ \ , a, b, c > 0, \tag{29}$$

and [15], we have

$$\begin{aligned} H_1(x) = {} & \frac{b^{\frac{a}{c}}}{\Gamma\left(1 + \frac{a}{c}\right)} \{-\ln[1 - F(x)]\}^{a-1} \\ & \times {}_1F_1\left(\frac{a}{c}; 1 + \frac{a}{c}; -b\{-\ln[1 - F(x)]\}^c\right), \end{aligned} \tag{30}$$

where $F(x)$ is given in (6).

The generalized gamma-generated Singh-Maddala distribution is obtained from (30) by substituting $p = h$:

$$\begin{aligned} H_1(x) = {} & b^{\frac{a}{c}}[(m-1)\ln(1 + \alpha x^p)]^{a-1} \\ & \times {}_1F_1\left(\frac{a}{c}; 1 + \frac{a}{c}; -b[(m-1)\ln(1 + \alpha x^p)]^c\right). \end{aligned} \tag{31}$$

Using another representation

$$H_2(x) = 1 - \frac{cb^{\frac{a}{c}}}{\Gamma\left(\frac{a}{c}\right)} \int_0^{-\ln(F(x))} w^{a-1} e^{-bw^c} dw, \quad a, b, c > 0, \tag{32}$$

and [15], we get

$$H_2(x) = 1 - \frac{b^{\frac{a}{c}}}{\Gamma\left(1 + \frac{a}{c}\right)} \{-\ln[F(x)]\}^a {}_1F_1\left(\frac{a}{c}; 1 + \frac{a}{c}; -b\{-\ln[F(x)]\}^c\right), \tag{33}$$

where $F(x)$ is given by (6).

8 Beta-generated GFSR-distribution

Using

$$G(x) = \frac{1}{B(\alpha_1, \beta_1)} \int_0^{F(x)} w^{\alpha_1 - 1}(1-w)^{\beta_1 - 1} dw, \quad \alpha_1, \beta_1 > 0, \tag{34}$$

and [1], we have

$$G(x) = \frac{F^{\alpha_1}(x)}{\alpha_1 B(\alpha_1, \beta_1)} {}_2F_1(\alpha_1, 1-\beta_1; 1+\alpha_1; F(x)), \tag{35}$$

where $F(x)$ is given in (6).

$G(x)$ may be rewritten as

$$G(x) = \frac{1}{\alpha_1 B(\alpha_1, \beta_1)} \sum_{r=0}^{\infty} \frac{(\alpha_1)_r (1-\beta_1)_r}{(1-\alpha_1)_r r!} F^{\alpha_1 + r}(x), \tag{36}$$

showing that $G(x)$ is an infinite linear combination of the distribution functions $F^{\alpha_1+r}(x)$.

It may be pointed out that [12] obtained a double series infinite linear combination given in their equation (8) for corresponding density function $g(x)$ in a particular case for Beta-generated Burr XII distribution when $p = h$.

For $p = h$, we have the following beta-generated distribution function from (35) for Singh-Maddala distribution:

$$G(x) = \frac{[1-(1+\alpha x^p)^{1-m}]^{\alpha_1}}{\alpha_1 B(\alpha_1, \beta_1)} {}_2F_1\big(\alpha_1, 1-\beta_1; 1+\alpha_1; 1-(1+\alpha x^p)^{1-m}\big). \tag{37}$$

Paraíba et al. [12] give a double infinite series involving distribution functions in their equation (10) for $G(x)$.

9 MLE and applications

Taking a random sample $\boldsymbol{x} = (x_1, ..., x_n)$ from $X \sim \text{GFSR}(\boldsymbol{\theta})$, $\boldsymbol{\theta} = (p, m, a, h)'$, the log-likelihood function is given by

$$\begin{aligned} \ln(L(\boldsymbol{\theta}|\boldsymbol{x})) = {} & \ln(f(\boldsymbol{x}|\boldsymbol{\theta})) = n\ln(k) + (p-1)\sum_{i=1}^{n} \ln(x_i) - m\sum_{i=1}^{n} \ln(1+\alpha x_i^h) \\ = {} & n\left[\ln(h) + \frac{p}{h}\ln(\alpha) - \ln\left(\Gamma\left(\frac{p}{h}\right)\right) - \ln\left(\Gamma\left(m - \frac{p}{h}\right)\right) + \ln\Gamma(m)\right] \\ & + (p-1)\sum_{i=1}^{n} \ln(x_i) - m\sum_{i=1}^{n} \ln(1+\alpha x_i^h). \end{aligned} \tag{38}$$

The estimation of p, m, α and h are obtained by solving numerically the following equations:

$$\frac{\partial \ln(L(\boldsymbol{\theta}|\boldsymbol{x}))}{\partial p} = n\left[\frac{\ln(a)}{h} - \frac{1}{h}\Psi\left(\frac{p}{h}\right) + \frac{1}{h}\Psi\left(m - \frac{p}{h}\right)\right] + \sum_{i=1}^{n} \ln(x_i) = 0,$$

$$\frac{\partial \ln(L(\boldsymbol{\theta}|\boldsymbol{x}))}{\partial m} = n\left[-\Psi\left(m - \frac{p}{h}\right) + \Psi(m)\right] - \sum_{i=1}^{n}\left[\ln(1 + ax_i^h)\right] = 0,$$

$$\frac{\partial \ln(L(\boldsymbol{\theta}|\boldsymbol{x}))}{\partial \alpha} = \frac{np}{ah} - m\sum_{i=1}^{n}\left[x_i^h/(1 + ax_i^h)\right] = 0,$$

$$\frac{\partial \ln(L(\boldsymbol{\theta}|\boldsymbol{x}))}{\partial h} = \frac{np}{h^2}\left[\frac{h}{p} - \ln(a) + \Psi\left(\frac{p}{h}\right) - \Psi\left(m - \frac{p}{h}\right)\right] - m\sum_{i=1}^{n}\frac{a\,{x_i}^h \ln(x_i)}{1 + a\,{x_i}^h} = 0,$$

where $\Psi(.)$ is the digamma function.

The model defined in (4) is applied to three real data sets given below:

(i) Air conditioning system: the failure times of the air conditioning system (see Table 1) of an airplane reported in [8]. The paper [16] analyzed this data.

Table 1: Failure times of the air conditioning system of an airplane.

1	5	11	11	14	14	16	21	42	52	71	87	95	120	246
3	7	11	12	14	16	20	23	47	62	71	90	120	225	261

(ii) Kevlar 49/Epoxy strands: the failure times of Kevlar 49/epoxy strands with pressure at 90% are given in Table 2. The failure times in hours were originally given in [3]. The papers [4], [2], and [11] analyzed this data.

Table 2: Failure times of Kevlar 49/epoxy strands.

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07
0.08	0.09	0.09	0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20
0.23	0.24	0.24	0.29	0.34	0.35	0.36	0.38	0.40	0.42	0.43	0.52
0.54	0.56	0.60	0.60	0.63	0.65	0.67	0.68	0.72	0.72	0.72	0.73
0.79	0.79	0.80	0.80	0.83	0.85	0.90	0.92	0.95	0.99	1.00	1.01
1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29	1.31	1.33
1.34	1.40	1.43	1.45	1.50	1.51	1.52	1.53	1.54	1.54	1.55	1.58
1.60	1.63	1.64	1.80	1.80	1.81	2.02	2.05	2.14	2.17	2.33	3.03
3.03	3.34	4.20	4.69	7.89							

Table 3: Remission times (in months) of bladder cancer patients.

0.08	0.20	0.40	0.50	0.51	0.81	0.90	1.05	1.19	1.26
1.35	1.40	1.46	1.76	2.02	2.02	2.07	2.09	2.23	2.26
2.46	2.54	2.62	2.64	2.69	2.69	2.75	2.83	2.87	3.02
3.25	3.31	3.36	3.36	3.48	3.52	3.57	3.64	3.70	3.82
3.88	4.18	4.23	4.26	4.33	4.34	4.40	4.50	4.51	4.87
4.98	5.06	5.09	5.17	5.32	5.32	5.34	5.41	5.41	5.49
5.62	5.71	5.85	6.25	6.54	6.76	6.93	6.94	6.97	7.09
7.26	7.28	7.32	7.39	7.59	7.62	7.63	7.66	7.87	7.93
8.26	8.37	8.53	8.65	8.66	9.02	9.22	9.47	9.74	10.06
10.34	10.66	10.75	11.25	11.64	11.79	11.98	12.02	12.03	12.07
12.63	13.11	13.29	13.80	14.24	14.76	14.77	14.83	15.96	16.62
17.12	17.14	17.36	18.10	19.13	20.28	21.73	22.69	23.63	25.74
25.82	26.31	32.15	34.26	36.66	43.01	46.12	79.05		

(iii) Bradder cancer patients: remission times (in months) of a random sample of 128 bladder cancer patients is given in [6] and analyzed by [7]. The data are given in Table 3.

The maximum likelihood method was used together with program R [14], through the function `cobyla` contained in the package `nloptr`, for the estimation of the parameters. The convergence of the maximization procedure involving generalized flexible distributions depends on the initial guesses. We used multiple starts based on the shape of the histogram of the data. The maximum likelihood estimates are given in the Table 4.

Table 4: Maximum likelihood estimates for GFSR model.

Data set	p	m	α	h
Airplane	1.9878	1.8149e+06	1.1506e-06	0.3101
Kevlar 49	0.6838	0.0632	3.8119e-10	51.2312
Bladder	1.3171	2.0631	0.0143	1.8289

The density and distribution function for histogram and empirical distribution function for each data set are given in the Figure 2.

We compare the GFSR model with: (i) gamma-Dagum (GD) model presented by [16], (ii) Dagum-Poisson (DP) model presented by [11], and (iii) McLomax model presented by [7].

(i) Gamma-Dagum (GD) model:

$$f(x) = \frac{\lambda\delta\beta^{\alpha}}{\Gamma(\alpha)} x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}\left[\ln(1+\lambda x^{-\delta})\right]^{\alpha-1}$$

and

$$F(x) = 1 - \frac{1}{\Gamma(\alpha)}\gamma\left(\alpha, \beta\ln(1+\lambda x^{-\delta})\right),$$

for $x > 0,\ \lambda,\ \delta,\ \beta > 0$, integer $\alpha > 0$, where

$$\gamma(\alpha, x) = \int_0^x t^{\alpha-1}\exp(-t)dt.$$

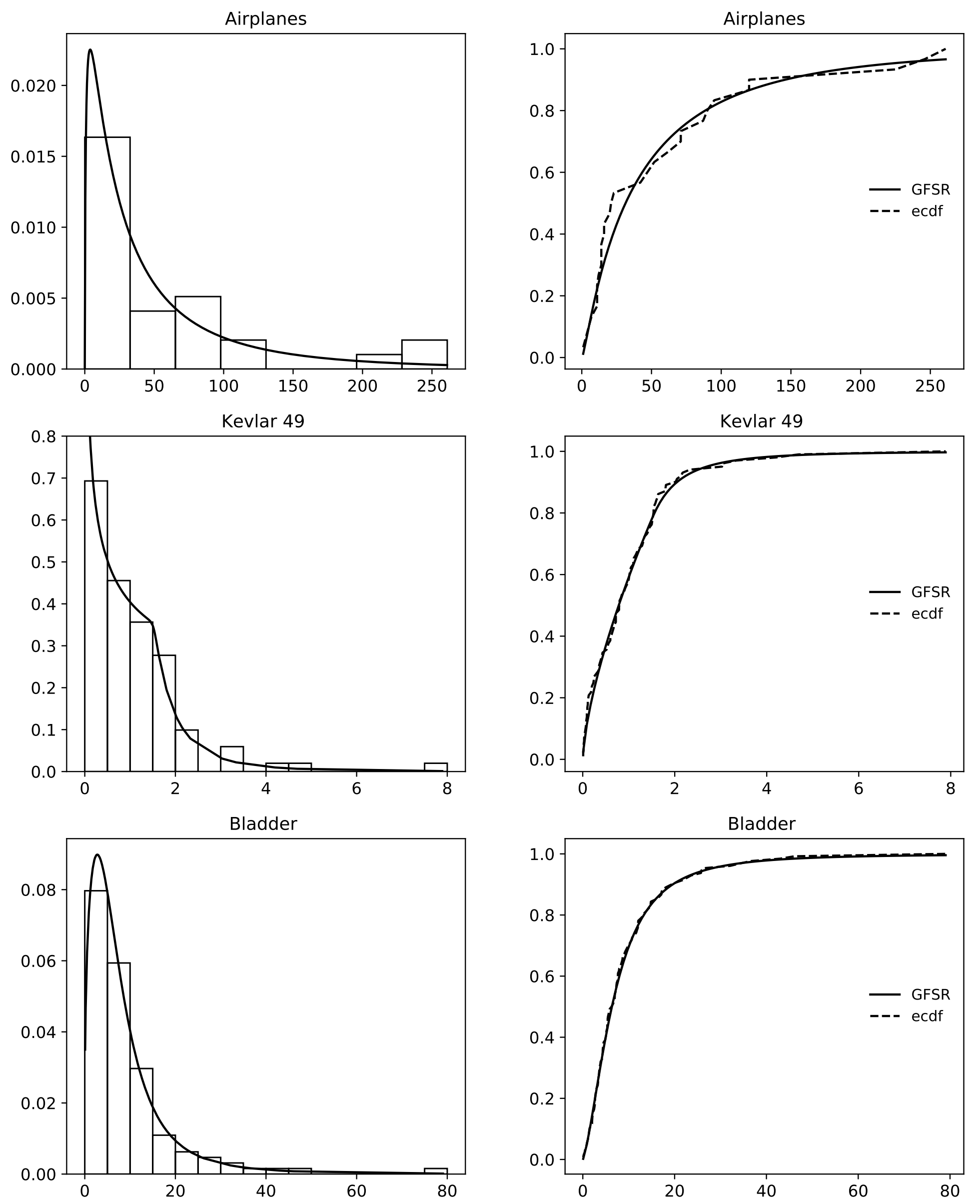

Figure 2: Adjusted GFSR distribution for the three data sets.

(ii) Dagum-Poisson (DP) model:

$$f(x)=\frac{\beta\lambda\delta\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\beta-1}\exp[\theta(1+\lambda x^{-\delta})^{-\beta}]}{\exp(\theta)-1}$$

and

$$F(x)=\frac{1-\exp[\theta(1+\lambda x^{-\delta})^{-\beta}]}{1-\exp(\theta)},$$

for $x>0,\ \lambda,\ \beta,\ \delta,\ \theta>0.$

(iii) McLomax model:

$$f(x)=\frac{c\alpha\beta^{\alpha}(\beta+x)^{-\alpha-1}}{B(ac^{-1},\eta+1)}\left[1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right]^{a-1}\left\{1-\left[1-\left(\frac{\beta}{\beta+x}\right)^{\alpha}\right]^{c}\right\}^{\eta},$$

for $x>0,\ a,\ c,\ \alpha,\ \beta>0, \eta\geq 0.$

We apply Kolmogorov-Smirnov (KS), Anderson-Darling (AD) and Cramér-von-Mises (CvM) statistics to assess the goodness of fit of the model. In general, the smaller the values of KS, AD, CvM, the better the fit (see [20]). The Akaike Information Criterion (AIC), corrected Akaike Information Criterion (AICc) and Bayesian Information Criterion (BIC) were used to compare the models. These criteria are defined by

$$\begin{aligned}\text{AIC}&=-2\ln(f(\boldsymbol{x}|\boldsymbol{\theta}))+2p,\\ \text{BIC}&=-2\ln(f(\boldsymbol{x}|\boldsymbol{\theta}))+p\log(n),\\ \text{AICc}&=-2\ln(f(\boldsymbol{x}|\boldsymbol{\theta}))+2p+\left[\frac{2p(p+1)}{n-p-1}\right],\end{aligned}$$

where $\ln(f(\boldsymbol{x}|\boldsymbol{\theta}))$ is the log-likelihood function, p the number of parameters of the model and n the sample size. The best model has the lowest value according to the criterion used. All the results are shown in Table 5.

Based on the statistics and selection criteria, we conclude that the GFSR model fits all the data set better than the other models. In addition, Figure 2 show that our proposed model is a good alternative for modeling the data.

10 Concluding remarks

The generalized-F-Shah-Rathie (GFSR) distribution defined in 1974 is studied in some detail. Some properties are given along with some new distributions obtained from GFSR-distribution. The GFSR-distribution is applied to analize three real data sets demonstrating its utility.

Table 5: Model selection criterion and goodness-of-fit statistic.

	Airplane		Kevlar 49		Bladder	
	GFSR	GD	GFSR	DP	GFSR	McLomax
AIC	310.68	311.16	204.18	208.09	827.31	829.82
AICc	312.28		204.60	208.51	827.63	
BIC	316.29		214.64	218.55	838.71	844.09
KS	0.0891		0.0406		0.0338	
AD	0.4272	0.4314	0.3123	0.4632	0.1111	0.1685
CvM	0.0791	0.0800	0.0409	0.0657	0.0171	0.0254

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Appendix - Distribution function

Using [9, p. 149]

$$(1+z)^a = \frac{1}{\Gamma(-a)} G^{1,1}_{1,1}\left[z \middle| \begin{matrix} 1+a \\ 0 \end{matrix} \right]$$

in (4), we have

$$\begin{aligned}
F(x) &= \int_0^x f(y)dy = \frac{k}{\Gamma(m)} \int_0^x y^{p-1} G^{1,1}_{1,1}\left[\alpha y^h \middle| \begin{matrix} 1-m \\ 0 \end{matrix} \right] dy \\
&= \frac{k}{\Gamma(m)} \frac{1}{\alpha^{\frac{p-1}{h}}} \int_0^x (\alpha y^h)^{\frac{p-1}{h}} G^{1,1}_{1,1}\left[\alpha y^h \middle| \begin{matrix} 1-m \\ 0 \end{matrix} \right] dy \\
&= \frac{k}{\Gamma(m)\alpha^{\frac{p-1}{h}}} \int_0^x G^{1,1}_{1,1}\left[\alpha y^h \middle| \begin{matrix} \frac{p-1}{h}+1-m \\ \frac{p-1}{h} \end{matrix} \right] dy \\
&\quad \text{(see [9])} \\
&= \frac{k}{\Gamma(m)\alpha^{\frac{p-1}{h}}} \frac{1}{2\pi i} \int_L \Gamma\left(\frac{p-1}{h} - s\right) \Gamma\left(m - \frac{p-1}{h} + s\right) \alpha^s \int_0^x y^{hs} ds dy \\
&\quad \text{(using (1))} \\
&= \frac{k}{\Gamma(m)\alpha^{\frac{p-1}{h}}} \frac{1}{2\pi i} \int_L \Gamma\left(\frac{p-1}{h} - s\right) \Gamma\left(m - \frac{p-1}{h} + s\right) \alpha^s \frac{x^{hs+1}}{h(s+\frac{1}{h})} ds, \\
&\quad \mathrm{Re}(hs+1) > 0,
\end{aligned}$$

$$
\begin{aligned}
=&\frac{kx}{h\Gamma(m)\alpha^{\frac{p-1}{h}}}\frac{1}{2\pi i}\int_L \Gamma\left(\frac{p-1}{h}-s\right)\frac{\Gamma\left(m-\frac{p-1}{h}+s\right)\Gamma\left(s+\frac{1}{h}\right)}{\Gamma\left(s+\frac{1}{h}+1\right)}(\alpha x^h)^s ds\\
=&\frac{kx}{h\Gamma(m)\alpha^{\frac{p-1}{h}}}G^{1,2}_{2,2}\left[\alpha x^h\middle|\begin{matrix}\frac{p-1}{h}+1-m,1-\frac{1}{h}\\ \frac{p-1}{h},-\frac{1}{h}\end{matrix}\right]\\
&\text{(using (1))}\\
=&\frac{k}{h\Gamma(m)\alpha^{\frac{p}{h}}}(\alpha x^h)^{\frac{1}{h}}G^{1,2}_{2,2}\left[\alpha x^h\middle|\begin{matrix}\frac{p-1}{h}+1-m,1-\frac{1}{h}\\ \frac{p-1}{h},-\frac{1}{h}\end{matrix}\right]\\
=&\frac{k}{h\Gamma(m)\alpha^{\frac{p}{h}}}G^{1,2}_{2,2}\left[\alpha x^h\middle|\begin{matrix}\frac{p}{h}+1-m,1\\ \frac{p}{h},0\end{matrix}\right]=\frac{1}{\Gamma\left(\frac{p}{h}\right)\Gamma\left(m-\frac{p}{h}\right)}G^{1,2}_{2,2}\left[\alpha x^h\middle|\begin{matrix}\frac{p}{h}+1-m,1\\ \frac{p}{h},0\end{matrix}\right],\\
&\text{(see [9])}
\end{aligned}
$$

where $x > 0$ and k is given in (5).

The contour integral can be written as a ${}_2F_1$, giving an alternative expression as

$$F(x) = \frac{\alpha^{\frac{p}{h}}x^p}{mB\left(\frac{p}{h}+1, m-\frac{p}{h}\right)}{}_2F_1\left(m, \frac{p}{h}; \frac{p}{h}+1; -\alpha x^h\right).$$

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