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Annuity interest rates and applications

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Abstract

The implicit expression for interest rate (R) involving principal loan amount (P), monthly payment (M) and time period in months (T) is studied for historical approximations, exact infinite series solution involving two arbitrary parameters, H-function solution, approximations including for large values of T in terms of infinite series as well as involving Lambert's W-function. A new approximation for R is obtained from the main result which resulted in better numerical values for R than that obtained by Cantrell (2007). Three numerical Tables for R are given in 7 Appendices along with mathematical derivations of several results derived in this paper.

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1. Introduction and historical background

The problem is to find monthly interest rate R knowing the monthly payment M for a principal loan amount P for a time period T months. The governing compound interest equation is:

$$M = \frac{P \cdot R}{1-(1+R)^{-T}} . \tag{1}$$

The equation (1) is valid when payments are made at the end of each month. If payments are made at the beginning of each month, then replace T by T-1 and P by P-M in (1), as the end of each month is the start of the next month.
Next, we briefly describe available interest rate approximations to (1).

(a) Simpson's approximation (Simpson (1767), p. 242):

$$R = \frac{30h - 4(2T+1)}{6h(5h-3T-4)+1/6\cdot(11T+13)(T+2)} \quad , \tag{2}$$

where

$$h = \frac{T(T+1)}{2(T-M/P)} . \tag{3}$$

(b) Baily's approximation (Baily (1808), p. 127):

$$R = h \cdot \left[\frac{12-(T-1)h}{12-2(T-1)h} \right], \tag{4}$$

where

$$h = \left(\frac{TM}{P} \right)^{\frac{2}{T+1}} - 1. \tag{5}$$

(c) M approximation (M, (1855)):

$$R = \frac{8(T-a)}{(T+1)(3a+T)}, \tag{6}$$

where

$$a = \frac{P}{M}. \tag{7}$$

(d) Henderson's approximation (Henderson (1907)):

$$R = h \cdot \left[\frac{T+1}{2T} + \frac{(T-1)h}{6+2(T-1)h} \right]^{-1}, \tag{8}$$

where

$$h = \frac{1}{a} - \frac{1}{T}. \tag{9}$$

with a defined in (7).

(e) Lenzi's approximation (Lenzi (1936)):

(Note: Lenzi reviewed the Baily equation, validating it for $T \leq 50$ and proposing the change below for $T>50$)

$$R = \frac{12h - 6k - (T-1)h^2}{6-(T-1)h}, \tag{10}$$

where

$$k = \left[\frac{Th}{1-(1+h)^{-T}} \right]^{\frac{2}{T+1}} - 1, \tag{11}$$

and

$$h = \left(\frac{TM}{P} \right)^{\frac{2}{T+1}} - 1. \tag{12}$$

(f) Evan's approximation (Evans (1946)):

$$R = \frac{2}{(T+1)p} \cdot \left[\left(\frac{TM}{P} \right)^p - 1 \right], \tag{13}$$

where

$$p=\frac{1}{5}+\frac{4}{3}\cdot\left[\frac{1}{(T+1)}+\frac{1}{10}\cdot\left(\frac{TM}{P}\right)^{1/2}\right]. \tag{14}$$

(g) Karpin's approximation (Karpin (1967)):

$$R=\left[\frac{2h(3+h)}{2T\cdot h+3(T+1)}\right], \tag{15}$$

where

$$h=\frac{TM}{P}-1. \tag{16}$$

(h) Cantrell's approximation (Cantrell (2007)):

$$R=\left[\left(1+\frac{M}{P}\right)^{1/q}-1\right]^{q}-1, \tag{17}$$

where

$$q=\frac{log\left(1+1/T\right)}{log(2)}. \tag{18}$$

(i) Fayed's approximation (Fayed (2011)):

$$R=\frac{1}{2(T-2)}\left[3-\left(9-\frac{24(T-2)(T-P/M)}{T(T-1)}\right)^{1/2}\right]. \tag{19}$$

It was observed that Cantrell's approximation is the best among all the approximations mentioned above. As Hawawini and Vora (1981) report, this problem has engaged mathematicians, actuaries and financial analysists for about three centuries.

The rest of the paper is divided as follow: Section 2 deals with the exact expressions for interest rate R in general as an infinite series as well as a H-function. The closed form results are given for T=2 and 3. In Section 3, approximate results are obtained, one of them involving Lambert's W-function. Section 4 deals with other approximate results, one of which is used to calculate the interest rate R resulting in better numerical Tables as compared to that given earlier by Cantrell (2007). The proofs of various results are given in Appendices 2 to 6. Appendix 1 includes Lagrange's inversion theorem and the definition of the H-function. Appendix 7 gives various Tables of numerical results involving the interest rate R comparing with the results derived from Cantrell (2007) approximation. The paper ends with a conclusion section and a list of references.

2. Exact expression for interest rate

In this section, we indicate closed-form solutions for interest rate Equation (1) in terms of an infinite series, using Lagrange's Method (see Theorem 1 below) along with other exact results. The proofs for easy results are omitted, and for the other results are given in Appendices.

Denoting m = M/P, Eq. (1) is written as

$$m = \frac{R}{1-(1+R)^{-T}} . \tag{20}$$

Note that R = m when T→∞.

Eq. (20) is written as

$$mT = \frac{RT}{1-(1+R)^{-T}} . \tag{21}$$

Considering y = RT, Eq. (21) changes to

$$mT = \frac{RT}{1-(1+y/T)^{-T}} . \tag{22}$$

For simple cases, Eq. (20) can be solved in closed form. For example, for T=1 month, solution of Eq. (20) yields

$$R = m - 1 . \tag{23}$$

For T = 2 months, Eq. (20) gives

$$R^2 + (2-m)R + 1 - 2m = 0 . \tag{24}$$

Solving Eq. (24) as a quadratic equation, we have

$$R = \frac{m - 2 + \sqrt{m(m+4)}}{2} . \tag{25}$$

For T = 3 months, Eq. (20) transforms to

$$R^3 + (3-m)R^2 + 3(1-m)R + 1 - 3m = 0 . \tag{26}$$

Solution of Eq. (26) is obtained as

$$R = 2\sqrt{-\frac{p}{3}}\, cosh\left[\frac{1}{3} ar\, cosh\left(-\frac{3|q|}{2p}\sqrt{-\frac{3}{p}}\right)\right] - 1 + \frac{m}{3} , \tag{27}$$

where

$$p = -\frac{m}{3}(m+3), \tag{28}$$

and

$$q = -\frac{m}{27}\left(2m^2 + 9m + 27\right). \tag{29}$$

Theorem 1 (see Appendix 2).

For B > 0 and α > 0, the following result is valid:

$$(B+R)^{\alpha} = (B+m)^{\alpha} - A(\alpha, T, m, B), \tag{30}$$

where

$$A(\alpha, T, m, B) = \alpha \sum_{s=1}^{\infty} \sum_{j=0}^{\infty} \frac{m^s (1-\alpha)_j}{s!\, j!} \frac{(1-B)^j \Gamma(sT+j-\alpha+s)(1+m)^{-(sT+j-\alpha+s)}}{\Gamma(sT+j-\alpha+1)}. \tag{31}$$

For B=1,

$$(1+R)^{\alpha} = (1+m)^{\alpha} - \alpha \sum_{s=1}^{\infty} \frac{m^s \Gamma(sT-\alpha+s)(1+m)^{-(sT-\alpha+s)}}{s!\, \Gamma(sT-\alpha+1)} = (1+m)^{\alpha} - A(\alpha, T, m, 1). \tag{32}$$

For α = 1, (32) yields

$$R = m - A(1, T, m, 1). \tag{33}$$

The result in terms of H-function (see Appendix 3) is

$$A(\alpha, T, m, 1) = m\alpha (1+m)^{\alpha - T - 1} H_{2,3}^{1,2}\left[\frac{m}{(1+m)^{T+1}} \;\middle|\; \begin{matrix} (0,1),(\alpha-T,T+1) \\ (0,1),(-1,1),(\alpha-T,T) \end{matrix} \right], \tag{34}$$

where H-function is defined in Appendix 1.

For the proof of the following result, see Appendix 4:

$$A(1,1,m,1) = 1. \tag{35}$$

The equation (35) implies that R = m-1.

3. Approximate expressions for interest rate

In this section, we obtain approximate solutions. The proofs of the results are given in Appendices.

Talking limit T→∞, Eq. (22) reduces to

$$mT = \frac{RT}{1-e^{-RT}}. \tag{36}$$

The solution of Eq. (36) using Lagrange series (see, Appendix 5) is

$$R = m - \frac{1}{T}\sum_{k=1}^{\infty}\frac{\left(mTke^{-mT}\right)^k}{k(k)!}. \tag{37}$$

On the other hand, the solution of (36) involving *W* function (see, Appendix 6) is

$$R = m + \frac{1}{T}W_0\left(-mTe^{-mT}\right), \tag{38}$$

where Lambert's W-function is given in Appendix 1.

4. Computation of approximate interest rates

Cantrell (2007) approximation with error less than 1%, is

$$R \cong \left[(1+m)^{1/q} - 1\right]^q - 1, \tag{39}$$

where

$$q = \log_2\left(1 + \frac{1}{T}\right). \tag{40}$$

Eq. (39) is rewritten as

$$(1+R)^{\frac{1}{q}} \cong (1+m)^{\frac{1}{q}} - 1. \tag{41}$$

Comparing equations (32) and (41), we have

$$\alpha = 1/q = 1/\log_2\left(1 + \frac{1}{T}\right), \tag{42}$$

and

$$\alpha\sum_{s=1}^{\infty}\frac{m^s}{s!}\frac{\Gamma(sT-\alpha+s)(1+m)^{-(sT-\alpha+s)}}{\Gamma(sT-\alpha+1)} = 1. \tag{43}$$

From (42) and (43), one gets

$$\sum_{s=1}^{\infty}\frac{m^s}{s!}\frac{\Gamma(sT-\alpha+s)(1+m)^{-(sT-\alpha+s)}}{\Gamma(sT-\alpha+1)} = \log_2\left(1 + \frac{1}{T}\right). \tag{44}$$

Several sets of values for B, α and A (α, T, m, B) are tried to reduce the error in approximate values of R. Talking B=1.000018, $\alpha=\log_2(1+1/T)$ and A (α, T, m, B)≅1 in (30), we get

$$R \cong \left[(1.000018 + m)^{1/log_2(1+1/T)} - 1\right]^{log_2(1+1/T)} - 1.000018, \tag{45}$$

resulting in a better result than what was indicated by Cantrell (2007).

The percentage error is calculated using the expression:

Error (%) = abs[(calculated rate - exact rate)/(exact rate)] x 100.

The numerical calculations using (45) along with Cantrell's (2007) approximation are given in Appendix 7, for comparison purposes.

Notations

The following symbols are used in this paper:

R = interest rate ($/$ per month);
M = monthly payment ($);
P = principal loan amount ($);
T = time period (months);
m = normalized monthly payment M/P;

5. Conclusions

The exact numerical results for R may be obtained from Eq. (30). However, the approximate expression for R given in Eq. (45) result in better numerical values as compared to that obtained earlier by Cantrell (2007). In addition, several new mathematical results involving interest rates are derived.

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References

Baily, F. (1808) The Doctrine of Interest and Annuities, Richard Taylor and Co., London.

Brown, S. (2016) Loan or investment formulas, https://brownmath.com/bsci/loan.htm,18/09/2016.

Cantrell, D.W. (2007) Re: Finding interest rate without approximation or root-finding? https://groups.google.com, 27/09/2007.

Evans, A.W. (1946) Further remarks on the relationship between the values of life annuities at different rates of interest, including a description of a method of first-difference interpolation and a reference to annuities-certain, J. of the Institute of Actuaries, 72(3), 447-454, Cambridge.

Fayed, K. A. (2011) Mathematical derivation of annuity rate and its application, International J. of Experimental Algorithms (IJEA), 2(2), 27-41.

Hawawini, G.A. and Vora, A. (1981) On the history of yield approximations, INSEAD, No. 81/14.

Henderson, R. (1907), On the determination of the rate of interest yielded by an investment, Transactions, Actuarial Society of America, Vol. 10, 659-666.

Karpin, H. (1967) Simple algebraic formulae for estimating the rate of interest, J. of the Institute of Actuaries, 93(2), 297-309, Cambridge.

Lenzi, E. (1936) Lezioni di Matematica Finanziaria, Napoli.

Mathai, A.M., Saxena, R.K. and Haubold, H.J. (2010) The H-function: Theory and Applications, Springer, New York.

Rathie, P.N. and Ozelim, L.C.S.M. (2012) General solutions to certain real degree equations and their applications, Advanced Studies in Contemporary Mathematics, 22(2), 325-341.

Simpson, T. (1767) A Treatise of Algebra", Ed. J. Nourse, 3rd Edition, London.

Whittaker, E.T. and Watson, G.N. (1990) A Course of Modern Analysis, Cambridge Univ. Press, 4th Edition, Cambridge.

Appendix 1 - Known results

(A) Lagrange's inversion theorem

Let y be defined as the following function of constant a, function ϕ, and a parameter θ

$$y = a + \theta\phi(y).$$

Then any function $f(y)$ is expressed as the following power series θ (Whittaker and Watson (1990))

$$f(y) = f(a) + \sum_{i=1}^{\infty} \frac{\theta^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} [f'(x)\phi^n(x)]|_{x=a}.$$

(B) H-function

The H-function is defined as

$$H_{p,q}^{m,n}\left[z\Big|_{(b_1,B_1),\dots,(b_m,B_m),(b_{m+1},B_{m+1}),\dots,(b_q,B_q)}^{(a_1,A_1),\dots,(a_n,A_n),(a_{n+1},A_{n+1}),\dots,(a_p,A_p)}\right]$$
$$= \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1-a_j+A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j s)\prod_{j=n+1}^{p}\Gamma(a_j - A_j s)} z^s ds.$$

where $z \neq 0$, an empty product is interpreted as unity, $0 \leq m \leq a$, $0 \leq n \leq p$ (not both m and n are zeros simultaneously). The parameters are such that no poles of $\prod_{i=1}^{m}\Gamma(b_j - B_j s)$ coincides with any pole of $\prod_{j=1}^{n}\Gamma(1-a_j+A_j s)$ and contour $L(\alpha - i\infty, \alpha + i\infty)$ separates these two types of poles. For more details, see (Mathai et al. (2010)).

(C) The single valued-function $W_0(x)$ as the solution of

$$W(x)e^{W(x)} = x,$$

for $x \geq \frac{1}{e}$ and $W(x) \geq 1$, is used in this article.

Appendix 2 - Theorem 1, proof of result (30)

The equation (20) is

$R = m - m(1+R)^{-T}$.

Using Lagrange's inversion theorem [eqs. (46) and (47)] for $f(y) = (\beta + m)^\alpha$, $\beta > 0$, $\alpha > 0$, we have

$$(B+R)^\alpha = (B+m)^\alpha + \alpha\sum_{s=1}^{\infty}\frac{(-m)^s}{s!}\frac{\partial^{s-1}}{\partial x^{s-1}}[(B+x)^{\alpha-1}(1+x)^{-Ts}]|_{x=m}$$

$$= (B+m)^\alpha + \alpha\sum_{s=1}^{\infty}\frac{(-m)^s}{s!}\sum_{j=0}^{\infty}\frac{(1-\alpha)_j(1-B)^j}{j!}\frac{\partial^{s-1}}{\partial x^{s-1}}\left[(1+x)^{\alpha-1-Ts-j}\right]|_{x=m}$$

$$= (B+m)^\alpha - \alpha\sum_{s=1}^{\infty}\sum_{j=0}^{\infty}\frac{m^s(1-\alpha)_j(1-B)^j\Gamma(Ts+j+s-\alpha)}{s!\,j!\,\Gamma(Ts+j+1-\alpha)}(1+m)^{-(Ts+j+s-\alpha)}$$

Another solution can be obtained by taking specific values of the parameters in (1) and (10) in an article by Rathie and Ozelim (2012).

Appendix 3 - Result in H-function (Proof of result (34))

We have, from (32)

$$A(\alpha,T,m,1)=\alpha(1+m)^{\alpha}\sum_{s=0}^{\infty}\frac{m^{s+1}\Gamma((s+1)T-\alpha+s+1)(1+m)^{-(s+1)(T+1)}}{(s+1)!\,\Gamma((s+1)T-\alpha+1)}$$

$$=m\alpha(1+m)^{\alpha-T-1}\sum_{s=0}^{\infty}\left(\frac{m}{(1+m)^{T+1}}\right)^{s}\frac{\Gamma(s+1)\Gamma(s(T+1)+T+1-\alpha)}{s!\,\Gamma(s+2)\Gamma(sT+T+1-\alpha)}$$

Comparing with H-function series (Mathai et al. (2010)), we have:
$h=1$, $b_h=0$, $B_h=1$, $m=1$, $z=m(1+m)^{-(T+1)}$, $s=3$, $p=2$, $n=2$, $b_2=-1$, $B_2=1$, $b_3=\alpha-T$, $B_3=T$, $a_1=0$, $A_1=1$, $a_2=\alpha-T$, $A_2=T+1$.

Hence

$$A(\alpha,T,m,1)=m\alpha(1+m)^{\alpha-T-1}H_{2,3}^{1,2}\left[\frac{m}{(1+m)^{T+1}}\Big|_{(0,1),(-1,1),(\alpha-T,\,T)}^{(0,1),(\alpha-T,\,T+1)}\right]$$

As T is a positive integer, the H-function may be written as a Meijer's G-function which is computable by using the softwares Mathematica or Maple.

Appendix 4 - Proof of result (35)

We take $B=1$ and $\alpha=\log_2(1+\frac{1}{T})=1$ for $T=1$ in (31) to get

$$A(1{,}1,m,1)=\sum_{s=1}^{\infty}\frac{m^{s}\Gamma(2s-1)(1+m)^{-(2s-1)}}{s!\,\Gamma(s)}.$$

Using the duplication formula,

$$\Gamma(2s-1)=2^{2s-2}\pi^{-\frac{1}{2}}\Gamma\left(s-\frac{1}{2}\right)\Gamma(s)$$

we have

$$A(1{,}1,m,1)=\frac{(1+m)}{4\pi^{\frac{1}{2}}}\sum_{s=1}^{\infty}\left[\frac{4m}{(1+m)^{2}}\right]^{s}\frac{\Gamma(s-\frac{1}{2})}{s!}$$

$$=\frac{(1+m)}{4\pi^{\frac{1}{2}}}\left\{\sum_{s=0}^{\infty}\left[\frac{4m}{(1+m)^{2}}\right]^{s}\frac{(-\frac{1}{2})_s}{s!}\Gamma(-\frac{1}{2})-\Gamma(-\frac{1}{2})\right\}$$

$$=\frac{(1+m)\Gamma(-\frac{1}{2})}{4\pi^{\frac{1}{2}}}\left[\ {}_1F_0(-\frac{1}{2};;\frac{4m}{(1+m)^{2}})-1\right]$$

$$=-\frac{(1+m)}{2}\left\{\left[1-\frac{4m}{(1+m)^{2}}\right]^{\frac{1}{2}}-1\right\}$$

$$= -\frac{(1+m)}{2}\left[\frac{m-1}{m+1} - 1\right] = 1$$

Cantrell (2007) did not provide any explication as to why he took the approximation to interest rate R as given in eq. (39). In our infinite series expression (30) for interest rate, α, B are arbitrary to choose. As $A(\alpha, T, m, B) = 1$ for $\alpha = 1$, $T = 1$ and $B = 1$, we approximate $A(\alpha, T, m, B)$ by 1. This type of approximation does not add to the error in calculation of R for large values of T. We take $\alpha = \log_2(1 + \frac{1}{T})$ so that $\alpha = 1$, for $T = 1$. Thus, the arbitrary parameter B may be choosen in such a way that the approximate interest rate became as close as possible to the exact interest rate.

Appendix 5 - Proof of (37)

Equation (36) is

$$R = m - m\exp(-RT).$$

Applying Lagrange's inversion theorem, we get

$$R = m + \sum_{k=1}^{\infty} \frac{(-m)^k}{k!} \frac{\partial^{k-1}}{\partial R^{k-1}} [\exp(-RTk)]|_{R=m}$$

$$= m - \frac{1}{T} \sum_{k=1}^{\infty} \frac{[mT\exp(-Tm)]^k k^{k-1}}{k!},$$

on simplification.

Appendix 6 - Proof of (38)

Equation (36) is rewritten as

$$R - m = -m\exp(-RT).$$

Multiplying both sides by $T\exp(-mT)$, we have

$$T(R - m)\exp(T(R - m)) = -mT\exp(-mT).$$

Hence, using the definition of Lambert's W-function, we have

$$T(R - m) = W_0(-mT\exp(-mT)).$$

Thus,

$$R = m + \frac{1}{T} W_0(-mT\exp(-mT)).$$

Appendix 7.

The following are the Tables with numerical values of the rate and respective percent errors in relation to the exact value.

Table 1: Cantrell Equation (39).

T	Exact Rate															
	0.005		0.01		0.02		0.03		0.04		0.05		0.075		0.10	
	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)
2	0.005000	0.00534	0.009999	0.01062	0.019996	0.02106	0.029991	0.03132	0.039983	0.04141	0.049974	0.05132	0.074943	0.07537	0.099902	0.09841
3	0.005000	0.00945	0.009998	0.01879	0.019993	0.03714	0.029983	0.05504	0.039971	0.07251	0.049955	0.08955	0.074902	0.13033	0.099831	0.16862
4	0.004999	0.01318	0.009997	0.02616	0.019990	0.05151	0.029977	0.07607	0.039960	0.09985	0.049939	0.12287	0.074867	0.17720	0.099773	0.22712
5	0.004999	0.01673	0.009997	0.03314	0.019987	0.06503	0.029971	0.09568	0.039950	0.12514	0.049923	0.15342	0.074836	0.21917	0.099722	0.27821
6	0.004999	0.02017	0.009996	0.03990	0.019984	0.07800	0.029966	0.11435	0.039940	0.14900	0.049909	0.18199	0.074807	0.25749	0.099676	0.32359
12	0.004998	0.03992	0.009992	0.07810	0.019970	0.14938	0.029936	0.21408	0.039891	0.27249	0.049838	0.32488	0.074676	0.43149	0.099493	0.50706
18	0.004997	0.05891	0.009989	0.11400	0.019957	0.21311	0.029911	0.29812	0.039852	0.36989	0.049785	0.42936	0.074602	0.53057	0.099424	0.57622
24	0.004996	0.07740	0.009985	0.14813	0.019946	0.27039	0.029889	0.36863	0.039822	0.44493	0.049749	0.50149	0.074571	0.57162	0.099432	0.56753
36	0.004994	0.11308	0.009979	0.21149	0.019927	0.36708	0.029858	0.47319	0.039785	0.53710	0.049717	0.56646	0.074598	0.53648	0.099567	0.43326
48	0.004993	0.14712	0.009973	0.26863	0.019912	0.44172	0.029839	0.53516	0.039773	0.56675	0.049723	0.55407	0.074682	0.42452	0.099728	0.27188
60	0.004991	0.17956	0.009968	0.31981	0.019901	0.49635	0.029831	0.56193	0.039780	0.55118	0.049752	0.49543	0.074774	0.30109	0.099849	0.15088
75	0.004989	0.21790	0.009962	0.37572	0.019892	0.53975	0.029833	0.55711	0.039803	0.49203	0.049803	0.39450	0.074868	0.17633	0.099935	0.06464
100	0.004986	0.27651	0.009955	0.45023	0.019888	0.56170	0.029853	0.48858	0.039858	0.35585	0.049884	0.23225	0.074954	0.06109	0.099987	0.01340
120	0.004984	0.31875	0.009951	0.49433	0.019891	0.54521	0.029878	0.40823	0.039899	0.25198	0.049931	0.13849	0.074982	0.02383	0.099996	0.00352
150	0.004981	0.37469	0.009946	0.53752	0.019903	0.48508	0.029914	0.28539	0.039946	0.13603	0.049971	0.05762	0.074996	0.00533	0.100000	0.00045
180	0.004979	0.42209	0.009944	0.55702	0.019919	0.40452	0.029945	0.18453	0.039973	0.06763	0.049989	0.02215	0.074999	0.00113	0.100000	0.00005
220	0.004976	0.47286	0.009945	0.55364	0.019941	0.29468	0.029972	0.09474	0.039990	0.02451	0.049997	0.00575	0.075000	0.00014	0.100000	0.00000
250	0.004975	0.50227	0.009947	0.53442	0.019955	0.22257	0.029984	0.05490	0.039996	0.01100	0.049999	0.00203	0.075000	0.00003	0.100000	0.00000
300	0.004973	0.53636	0.009952	0.48153	0.019974	0.13112	0.029994	0.02085	0.039999	0.00276	0.050000	0.00035	0.075000	0.00000	0.100000	0.00000
360	0.004972	0.55574	0.009960	0.40077	0.019987	0.06455	0.029998	0.00613	0.040000	0.00050	0.050000	0.00004	0.075000	0.00000	0.100000	0.00000
AVERAGE ERROR =		0.21790	-	0.28270	-	0.27200	-	0.24620	-	0.22863	-	0.21620	-	0.19616	-	0.18424

Table 2: Proposed Equation (45).

T	Exact Rate															
	0.005		0.01		0.02		0.03		0.04		0.05		0.075		0.10	
	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)	R	Error (%)
2	0.005006	0.11382	0.010005	0.04854	0.020002	0.00810	0.029996	0.01214	0.039989	0.02722	0.049980	0.04013	0.074949	0.06816	0.099907	0.09317
3	0.005008	0.16865	0.010007	0.06933	0.020001	0.00602	0.029992	0.02686	0.039979	0.05180	0.049963	0.07332	0.074910	0.12004	0.099839	0.16128
4	0.005010	0.19981	0.010008	0.07886	0.020000	0.00044	0.029987	0.04295	0.039970	0.07568	0.049948	0.10405	0.074876	0.16546	0.099781	0.21888
5	0.005011	0.21910	0.010008	0.08274	0.019998	0.00905	0.029982	0.05962	0.039960	0.09900	0.049933	0.13320	0.074845	0.20677	0.099730	0.26964
6	0.005012	0.23163	0.010008	0.08341	0.019996	0.01885	0.029977	0.07651	0.039951	0.12175	0.049919	0.16105	0.074816	0.24485	0.099685	0.31499
12	0.005013	0.25222	0.010006	0.06204	0.019983	0.08485	0.029948	0.17444	0.039902	0.24507	0.049848	0.30463	0.074685	0.42041	0.099500	0.50022
18	0.005012	0.24364	0.010003	0.02816	0.019970	0.15027	0.029922	0.26104	0.039862	0.34525	0.049794	0.41188	0.074609	0.52189	0.099429	0.57135
24	0.005011	0.22729	0.009999	0.00787	0.019958	0.21087	0.029900	0.33490	0.039831	0.42339	0.049757	0.48681	0.074576	0.56501	0.099436	0.56417
36	0.005009	0.18745	0.009992	0.07872	0.019937	0.31515	0.029866	0.44604	0.039792	0.52110	0.049722	0.55639	0.074600	0.53276	0.099568	0.43169
48	0.005007	0.14522	0.009986	0.14468	0.019921	0.39703	0.029846	0.51361	0.039778	0.55503	0.049726	0.54726	0.074683	0.42246	0.099729	0.27117
60	0.005005	0.10325	0.009980	0.20471	0.019908	0.45810	0.029837	0.54492	0.039783	0.54265	0.049755	0.49085	0.074775	0.29995	0.099849	0.15055
75	0.005003	0.05241	0.009973	0.27121	0.019898	0.50837	0.029837	0.54450	0.039805	0.48631	0.049804	0.39173	0.074868	0.17579	0.099935	0.06452
100	0.004999	0.02689	0.009964	0.36160	0.019892	0.53925	0.029856	0.48097	0.039859	0.35294	0.049884	0.23106	0.074954	0.06094	0.099987	0.01337
120	0.004996	0.08500	0.009958	0.41680	0.019894	0.52809	0.029879	0.40317	0.039900	0.25028	0.049931	0.13788	0.074982	0.02377	0.099996	0.00351
150	0.004992	0.16319	0.009953	0.47421	0.019905	0.47369	0.029915	0.28264	0.039946	0.13527	0.049971	0.05740	0.074996	0.00531	0.100000	0.00045
180	0.004988	0.23092	0.009949	0.50537	0.019921	0.39696	0.029945	0.18304	0.039973	0.06730	0.049989	0.02207	0.074999	0.00112	0.100000	0.00005
220	0.004985	0.30594	0.009949	0.51432	0.019942	0.29031	0.029972	0.09408	0.039990	0.02440	0.049997	0.00574	0.075000	0.00014	0.100000	0.00000
250	0.004982	0.35156	0.009950	0.50240	0.019956	0.21968	0.029984	0.05454	0.039996	0.01095	0.049999	0.00202	0.075000	0.00003	0.100000	0.00000
300	0.004980	0.40930	0.009954	0.45882	0.019974	0.12967	0.029994	0.02072	0.039999	0.00275	0.050000	0.00034	0.075000	0.00000	0.100000	0.00000
360	0.004977	0.45227	0.009961	0.38574	0.019987	0.06391	0.029998	0.00609	0.040000	0.00050	0.050000	0.00004	0.075000	0.00000	0.100000	0.00000
AVERAGE ERROR =		0.20848	-	0.23898	-	0.24043	-	0.22814	-	0.21693	-	0.20789	-	0.19174	-	0.18145

Table 3: Comparison of Errors.

Exact Rate	Average Error (%)	
	Cantrell Equation	Proposed Equation
0.005	0.21790	0.20848
0,010	0.28270	0.23898
0.020	0.27200	0.24043
0.030	0.24620	0.22814
0.040	0.22863	0.21693
0.050	0.21620	0.20789
0.075	0.19616	0.19174
0.100	0.18424	0.18145
Global Average Error (%) =	0.23050	0.21426

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