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Minimal matrix representations for six-dimensional nilpotent Lie algebras

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Abstract

This paper is concerned with finding minimal dimension linear representations for six-dimensional real, indecomposable nilpotent Lie algebras. It is known that all such Lie algebras can be represented in $\mathfrak{gl}(6,\mathbb{R})$. After discussing the classification of the 24 such Lie algebras, it is shown that only one algebra can be represented in $\mathfrak{gl}(4,\mathbb{R})$. A Theorem is then presented that shows that 13 of the algebras can be represented in $\mathfrak{gl}(5,\mathbb{R})$. The special case of filiform Lie algebras is considered, of which there are five, and it is shown that each of them can be represented in $\mathfrak{gl}(6,\mathbb{R})$ and not $\mathfrak{gl}(5,\mathbb{R})$. Of the remaining five algebras, four of them can be represented minimally in $\mathfrak{gl}(5,\mathbb{R})$. That leaves one difficult case that is treated in detail in an Appendix.

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1 introduction

This paper forms part of a series whose goal is to find linear representations of minimal dimension for all the Lie algebras of low dimensions. In a previous work we have succeeded in finding linear representations for all indecomposable Lie algebras of dimension five and less [5] and very recently we have been able to carry out the same program for decomposable algebras of dimension five and less [7]. We have also extended our work to Lie algebras that admit a non-trivial Levi decomposition up to and including dimension eight [6]. Several other authors have investigated the problem of finding minimal dimensional representations of Lie algebras, see [2, 9] for example.

Of course it is an interesting and challenging mathematical problem to find such minimal dimension representations but there are also compelling practical reasons. Besides the value of having explicit representations of low-dimensional Lie algebras, they also add to the growing body of results that seek to provide alternatives to Ado's Theorem for the construction of representations, see [3]. Although Ado's theorem guarantees the existence of a matrix representation, it is of no practical utility in constructing them and certainly not helpful in finding representations of minimal dimension. Calculations involving symbolic programs such as Maple and Mathematica use up lots of memory when storing matrices; accordingly, calculations are likely to be faster if one can represent matrix Lie algebras using matrices of a small size. On the downside, it is true that choosing a smaller size representation may entail using more complicated entries in the representing matrices; one can see this phenomenon even more clearly if one constructs a matrix Lie group that gives rise to a matrix representation of the algebra and an attendant set of invariant vector fields or one-forms. One cannot have simultaneous simplicity in all aspects of a Lie algebra or Lie group representation and so various choices among such representations have to be made according to the kind of application one has in mind.

In this paper we are concerned with finding linear representations for real indecomposable nilpotent Lie algebras in dimension six that are of minimal dimension. It is convenient to define an invariant $\mu(\mathfrak{g})$ for a Lie algebra $\mathfrak{g}$ to be the dimension of a minimal dimensional representation of $\mathfrak{g}$. It is known from previous work [4] that for all the indecomposable six-dimensional nilpotent Lie algebras $\mu \leq 6$. We adopt the numbering given in [10] and [12]. Refer also to Section 2 below for a refinement in this classification.

In this article we give a Lie *group* corresponding to each of the 24 six-dimensional nilpotent Lie algebras that is a subgroup of $GL(4, \mathbb{R})$, $GL(5, \mathbb{R})$ or $GL(6, \mathbb{R})$, respectively. The representation for the Lie algebra is then easily obtained by differentiating and evaluating at the identity. In order to ensure that we have a bona fide group representation we provide also a list of right-invariant vector fields in each case.

An outline of this paper is as follows. In Section 2 we give a brief description of the indecomposable six-dimensional nilpotent Lie algebras. Since we know that for the indecomposable six-dimensional nilpotent Lie algebras $\mu \leq 6$, in