Mathematica Aeterna, Vol. 8, 2018, no. 3, 159 - 163

The Functor F-Tor in Category of L-fuzzy R-modules

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Abstract

In this paper, we show that every L-fuzzy R-module has an L-fuzzy projective resolution. Furthermore, we define the notions of functors F-Tor, and discuss its properties.

Mathematics Subject Classification: 08A72, 16D90, 16E05, 16E30

Keywords: L-fuzzy R-module, F-Tor, L-fuzzy projective resolution.

1 Introduction

Since Rosenfield [8] introduced fuzzy groups, many researchers are engaged in extending the concepts and results of abstract algebra to the broader framework of the fuzzy setting. Negoita and Ralescu [6] introduced the notion of fuzzy modules. Pan [7] introduced the category of fuzzy modules. Zahedi and Ameri [1, 11]defined fuzzy projective and injective presentations of a fuzzy R-module. By using these definitions three functors F-ext, F-$\overline{ext}$ and F-tor are presented.

The purpose of this paper is to study the functors F-Tor, we obtain that every L-fuzzy R-module has an L-fuzzy projective resolution.

Throughout this paper R is a commutative ring with identity, and L is a completely heyting algebra, which has the least and greatest elements, denoting 0 and 1 respectively. Then an L-fuzzy set μ in X is characterised by a map $\mu : X \to L$[2].

2 Preliminaries

Definition 2.1. *([7], Definition 1.1)* Let A be an R-module. A function μ_A is called L-fuzzy R-module if the map μ from A to L, satisfies (1) $\mu(x+y) \geq \wedge\{\mu(x), \mu(y)\}$, (2) $\mu(rx) \geq \mu(x)$, (3) $\mu(0) = 1$, for all $x, y \in A$ and $r \in R$.

Definition 2.2. *([7], Definition 1.1)* Let μ_A and η_B be L-fuzzy R-modules, a function $\tilde{f}:\mu_A\to\eta_B$ is called an F-homomorphism if $f:A\to B$ is an R-homomorphism and $\eta(f(x))\geq\mu(x)$ for all $x\in A$.

Lemma 2.3. *([7], Lemma 1.1) The category R-mod(L) of L-fuzzy R-modules is constituted, as follows:*

(1) $Obj(R\text{-mod}(L))$ is the family of all L-fuzzy R-modules.

(2) For any $\mu_A,\eta_B\in Obj(R-mod(L))$, the set of homomorphisms is

$$Hom_R(\mu_A,\eta_B)=\{\tilde{f}|\tilde{f}:\mu_A\to\eta_B \text{ is an } F\text{–homomorphism}\}.$$

The composition of homomorphisms is the usual composition of maps.

Definition 2.4. Let $\tilde{f}:\mu_A\to\eta_B$ be an F-homomorphism. $\eta_{Im\tilde{f}}$ is called the image of $\tilde{f}$, denoted by $Im\tilde{f}$. Further, μ_{V_0} is called the kernel of $\tilde{f}$, denoted by $Ker\tilde{f}$, *where* $V_0=\{x\in A|\eta(f(x))=1\}$.

Definition 2.5. *([10], Definition 3.1)* (i) A sequence $\cdots\to\mu_{n-1_{A_{n-1}}}\xrightarrow{\tilde{f}_{n-1}}\mu_{n_{A_n}}\xrightarrow{\tilde{f}_n}\mu_{n+1_{A_{n+1}}}\to\cdots$ of an F-homomorphism is said to be L-fuzzy exact if and only if $Im\tilde{f}_{n-1}=Ker\tilde{f}_n$ for all n.

(ii) An L-fuzzy exact sequence $\bar{0}\to\rho_C\xrightarrow{\tilde{f}}\eta_B\xrightarrow{\tilde{g}}\mu_A\to\bar{0}$ is called an L-fuzzy short exact sequence.

Theorem 2.6. *([10], Theorem 3.17) Let $\bar{0}\to\rho_{B'}\xrightarrow{\tilde{f}}\eta_B\xrightarrow{\tilde{g}}\gamma_{B''}$ be an L-fuzzy exact sequence and μ_A an arbitrary L-fuzzy module. The induces sequence $\bar{0}\to Hom_R(\mu_A,\rho_{B'})\xrightarrow{\tilde{f}_*}Hom_R(\mu_A,\eta_B)\xrightarrow{\tilde{g}_*}Hom_R(\mu_A,\gamma_{B''})$ is exact if and only if for any $\phi\in Hom_R(A,B')$, where $f_*\phi=\psi$, $\tilde{\psi}\in Ker\tilde{g}_*$, one has $\phi^{-1}\cdot\rho\geq\mu$.*

Definition 2.7. *([3], Definition 3.1)* A nonzero R-homomorphism $f:A\to B$ is called an admissible R-homomorphism if $\tilde{f}:\mu_A\to\eta_B$ is an F-homomorphism.

Definition 2.8. *([11], Definition 4.1)* An L-fuzzy short exact sequence $\bar{0}\to\rho_C\xrightarrow{\tilde{f}}\eta_P\xrightarrow{\tilde{g}}\mu_A\to\bar{0}$ of L-fuzzy R-modules with an L-fuzzy projective R-module η_P, is called an L-fuzzy projective presentation of μ_A. Moreover, it has the following form $\bar{0}\to\mu_{0_C}\xrightarrow{\tilde{\alpha}}\mu_{0_P}\xrightarrow{\tilde{\beta}}\mu_A\to\bar{0}$, where $\mu_0=\chi_{\{0\}}$.

Definition 2.9. *([1], Definition 2.6)* For any $\mu_A,\eta_B\in Obj(R\text{-mod}(L))$ and to the L-fuzzy projective presentation $\bar{0}\to\mu_{0_C}\xrightarrow{\tilde{f}}\mu_{0_P}\xrightarrow{\tilde{g}}\mu_A\to\bar{0}$ of μ_A. We define

$$F\text{–}tor_R^{\tilde{g}}(\mu_A,\eta_B)=Ker(\tilde{f}_*=\tilde{f}\otimes 1_{\eta_B}:\mu_{0_C}\otimes_R\eta_B\to\mu_{0_P}\otimes_R\eta_B).$$

3 Main Results

Definition 3.1. For any $\mu_A, \eta_B \in Obj(R\text{-mod}(L))$ and to the L-fuzzy projective presentation $\bar{0} \to \eta_{0_C} \xrightarrow{\tilde{f}} \eta_{0_P} \xrightarrow{\tilde{g}} \eta_B \to \bar{0}$ of η_B. We define

$$F\text{–}Tor_R^{\tilde{g}}(\mu_A, \eta_B) = Ker(1_{\mu_A} \otimes \tilde{f} : \mu_A \otimes_R \eta_{0_C} \to \mu_A \otimes_R \eta_{0_P}).$$

Theorem 3.2. *For any $\mu_A, \eta_B \in Obj(R\text{-mod}(L))$ and to the L-fuzzy projective presentation $\bar{0} \to \mu_{0_C} \xrightarrow{\tilde{f}} \mu_{0_P} \xrightarrow{\tilde{g}} \mu_A \to \bar{0}$ of μ_A, $\bar{0} \to \eta_{0_D} \xrightarrow{\tilde{k}} \eta_{0_{P'}} \xrightarrow{\tilde{l}} \eta_B \to \bar{0}$ of η_B. If all $\psi \in Hom_R(A, B)$ are admissible R-homomorphisms, then there is an isomorphism*

$$\alpha : F\text{–}Tor_R^{\tilde{l}}(\mu_A, \eta_B) \cong F\text{–}tor_R^{\tilde{g}}(\mu_A, \eta_B)$$

Proof. By the third chapter 8th section of [4] there is an isomorphism $\beta : Tor_R^{\tilde{l}}(\mu_A, \eta_B) \cong tor_R^{\tilde{g}}(\mu_A, \eta_B)$.

We show that β induces the isomorphism α. Let $\beta[\varphi] = [\psi]$, where $\varphi : C \to D$ and $\psi : A \to B$ are R-module homomorphisms. It is clear that $\tilde{\varphi} : \mu_{0_C} \to \eta_{0_D}$. Since $\psi \in Hom_R(A, B)$, we have $\tilde{\psi} : \mu_A \to \eta_B$ is an F-homomorphism.

Hence, the map α satisfies $\alpha[\tilde{\varphi}] = [\tilde{\psi}]$.

Definition 3.3. An L-fuzzy projective resolution of $\mu_A \in Obj(R\text{-mod}(L))$ is an exact sequence $\Theta_P : \cdots \to \theta_{2_{P_2}} \xrightarrow{\tilde{d}_2} \theta_{1_{P_1}} \xrightarrow{\tilde{d}_1} \theta_{0_{P_0}} \xrightarrow{\tilde{\varepsilon}} \mu_A \to \bar{0}$, in which each $\theta_{n_{P_n}}$ is an L-fuzzy projective R-module.

If Θ_P is an L-fuzzy projective resolution of μ_A, then its deleted L-fuzzy projective resolution is the L-fuzzy complex $\Theta_{P_{\mu_A}} : \cdots \to \theta_{2_{P_2}} \xrightarrow{\tilde{d}_2} \theta_{1_{P_1}} \xrightarrow{\tilde{d}_1} \theta_{0_{P_0}} \to \bar{0}$.

Theorem 3.4. *For any $\mu_A \in Obj(R\text{-mod}(L))$, μ_A has an L-fuzzy projective resolution.*

Proof. There is an L-fuzzy projective module $\theta_{0_{P_0}}$ and exact sequence $\bar{0} \to \theta_{0_{K_0}} \xrightarrow{\tilde{i}_0} \theta_{0_{P_0}} \xrightarrow{\tilde{\varepsilon}} \mu_A \to \bar{0}$, where $\theta_{0_{K_0}} = Ker\tilde{\varepsilon}$. Definition 2.8 implies that it has the following form $\bar{0} \to \mu_{0_{K_0}} \xrightarrow{\tilde{i}_0} \mu_{0_{P_0}} \xrightarrow{\tilde{\varepsilon}} \mu_A \to \bar{0}$, where $\mu_0 = \chi_{\{0\}}$.

Similarly, there is an L-fuzzy projective module $\theta_{1_{P_1}}$, a surjection $\tilde{\varepsilon}_1 : \theta_{1_{P_1}} \to \mu_{0_{K_0}}$ and an exact sequence $\bar{0} \to \theta_{1_{K_1}} \xrightarrow{\tilde{i}_1} \theta_{1_{P_1}} \xrightarrow{\tilde{\varepsilon}_1} \mu_{0_{K_0}} \to \bar{0}$. Splicing these together, we can define $\tilde{d}_1 : \mu_{0_{P_1}} \to \mu_{0_{P_0}}$ to be the composite $\tilde{i}_0 \cdot \tilde{\varepsilon}_1$. This construction can be iterated for all $n \geq 0$ and the ultimate exact sequence is infinitely long. An L-fuzzy projective resolution of μ_A has the following form $\cdots \to \mu_{0_{P_2}} \xrightarrow{\tilde{d}_2} \mu_{0_{P_1}} \xrightarrow{\tilde{d}_1} \mu_{0_{P_0}} \xrightarrow{\tilde{\varepsilon}} \mu_A \to \bar{0}$. The result holds. □

Definition 3.5. Let $T = -\otimes_R \eta_B$, and $F\text{–}Tor_n^R(-, \eta_B) = L^n T$. If an L-fuzzy projective resolution of μ_A is $\Theta_P : \cdots \to \mu_{0_{P_2}} \xrightarrow{\tilde{d}_2} \mu_{0_{P_1}} \xrightarrow{\tilde{d}_1} \mu_{0_{P_0}} \to \mu_A \to \bar{0}$, then

$$F\text{–}Tor_n^R(\mu_A, \eta_B) = H_n(\Theta_{P_{\mu_A}} \otimes_R \eta_B) = Ker(\tilde{d}_n \otimes 1_{\eta_B})/Im(\tilde{d}_{n+1} \otimes 1_{\eta_B}),$$

where $\tilde{d}_n \otimes 1_{\eta_B} \in Hom_{R(L)}(\mu_{0_{P_n}} \otimes \eta_B, \mu_{0_{P_{n+1}}} \otimes \eta_B)$.

Theorem 3.6. *If μ_A is an L-fuzzy projective R-module, then $F\text{–}Tor_n^R(\mu_A, \eta_B) = 0$ for all L-fuzzy R-module η_B and all $n \geq 1$.*

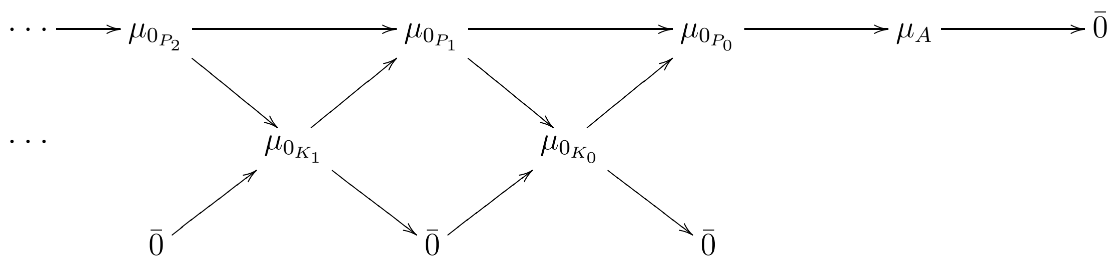
Proof. Since μ_A is an L-fuzzy projective module, $\cdots \to \bar{0} \to \bar{0} \xrightarrow{\tilde{d}_1} \mu_{0_{P_0}}(= \mu_{0_A}) \xrightarrow{\tilde{\varepsilon}} \mu_A \to \bar{0}$ is an L-fuzzy projective resolution of μ_A.
$\cdots \to 0 \to 0 \xrightarrow{\tilde{d}_1 \otimes 1_{\eta_B}} \mu_{0_A} \otimes_R \eta_B \xrightarrow{\tilde{d}_0 \otimes 1_{\eta_B}} 0$ is a chain complex, therefore $F-Tor_n^R(\mu_A, \eta_B) = 0 (n \geq 1)$. □

Remark 3.7. *Refer to Theorem 4.5 of [11] and Corollary 6.21 of [9], we can get that $F\text{-}Tor_R^n(\mu_A, \eta_B)$ does not depend on the chosen L-fuzzy projective resolution of μ_A.*

Theorem 3.8. *Let $\cdots \to \mu_{0_{P_2}} \xrightarrow{\tilde{d}_2} \mu_{0_{P_1}} \xrightarrow{\tilde{d}_1} \mu_{0_{P_0}} \xrightarrow{\tilde{\varepsilon}} \mu_A \to \bar{0}$ be an L-fuzzy injective resolution of μ_A, define $\mu_{0_{K_0}} = Ker\tilde{\varepsilon}$ and $\mu_{0_{K_n}} = Ker\tilde{d}_n$ for $n \geq 1$, we have*

$$F\text{–}Tor_{n+1}^R(\mu_A, \eta_B) \cong F\text{–}Tor_n^R(\mu_{K_0}, \eta_B) \cong \cdots \cong F\text{–}Tor_1^R(\mu_{K_{n-1}}, \eta_B).$$

Proof. Let



Since $\cdots \to \mu_{0_{P_2}} \xrightarrow{\tilde{d}_2} \mu_{0_{P_1}} \xrightarrow{\tilde{d}_1} \mu_{0_{K_0}} \to \bar{0}$ is an L-fuzzy projective resolution of $\mu_{0_{K_0}}$, we have the complex $\cdots \to \mu_{0_{P_2}} \otimes \eta_B \xrightarrow{\tilde{d}_2 \otimes 1_{\eta_B}} \mu_{0_{P_1}} \otimes \eta_B \xrightarrow{\tilde{d}_1 \otimes 1_{\eta_B}} \bar{0}$. By Definition 3.5, we have $F\text{–}Tor_n^R(\mu_{K_0}, \eta_B) = Ker(\tilde{d}_{n+1} \otimes 1_{\eta_B})/Im(\tilde{d}_{n+2} \otimes 1_{\eta_B}) = F\text{–}Tor_{n+1}^R(\mu_A, \eta_B)$. Similarly, $\cdots \to \mu_{0_{P_3}} \xrightarrow{\tilde{d}_3} \mu_{0_{P_2}} \xrightarrow{\tilde{d}_2} \mu_{0_{K_1}} \to \bar{0}$ is an L-fuzzy projective resolution of $\mu_{0_{K_1}}$, then

$$F\text{–}Tor_{n-1}^R(\mu_{K_1}, \eta_B) = Ker(\tilde{d}_{n+1} \otimes 1_{\eta_B})/Im(\tilde{d}_{n+2} \otimes 1_{\eta_B}) = F\text{–}Tor_{n+1}^R(\mu_A, \eta_B).$$

The remaining isomorphisms are obtained by iteration. □

ACKNOWLEDGEMENTS. This research is supported by the Fund of the Key Disciplines in the General Colleges and Universities of Xin Jiang Uygur Autonomous Region(XJEDU2016S080).

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Received: May 05, 2018