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Surfaces of Arbitrary Constant Negative Gaussian Curvature and Related Sine-Gordon Equations

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Abstract

Surfaces of arbitrary constant negative Gaussian curvature are investigated using the fundamental equations of surface theory and the notion of line congruences. It is shown that such surfaces can be generated by means of solutions to a particular form of sine-Gordon equation. A Bäcklund transformation is found for this equation and it is shown how this can be used to construct nontrivial solutions to it. The theorem of permutability is formulated for the system as well.

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1 Introduction

A great deal of the evolution of ideas pertaining to nonlinear equations, surfaces and solitons had their origins in investigations concerning the sine-Gordon equation [**1, 2**]. The study of surfaces with constant Gaussian curvature dates back to E. Bour [**3**], who in 1862 generated the particular form

$$\omega_{uv} = \frac{1}{\rho^2} \sin(\omega), \tag{1.1}$$

where $K = -1/\rho^2$. Although much has been written with regard to this system, it seems to invariably return to the case in which $K = -1$. Surfaces with constant Gaussian curvature are of great interest [**4, 5**]. The intention here is to develop an equation similar in form to (1.1) of sine-Gordon type such that K for the corresponding surface is negative but arbitrary. Such an equation will be referred to as a deformed sine-Gordon equation and the discussion can be thought to pertain to any two-dimensional manifold which can be embedded in $\mathbb{R}^3$. In fact, any compact, smooth two-manifold can be embedded smoothly in $\mathbb{R}^3$. This enables the use of the natural metric $\langle , \rangle$ on $\mathbb{R}^3$ so that lengths can be calculated as well as angles between normals in order that the formalism of a line congruence can be invoked. A two parameter family of lines in $\mathbb{R}^3$ or $\mathbb{R}^{2+1}$ forms a line congruence, and all normal lines of a surface form a line congruence called a normal line congruence. A line congruence can be expressed by writing $\mathbf{Y} = \mathbf{X}(u, v) + \lambda \mathbf{q}(u, v)$, $\langle \mathbf{q} \, , \mathbf{q} \rangle = 1$. For fixed parameters u, v, this represents a straight line passing through $\mathbf{X}(u, v)$ in the direction $\mathbf{q}(u, v)$. This then is a two parameter family of straight lines, or a line congruence. This idea appears in a formulation of Bäcklund's theorem which will be invoked to aid in establishing the claims which are formulated, as well as the fundamental equations for a two-manifold or surface.

Suppose that S and S^* are two focal surfaces of a line congruence, and PP^* is the line in the congruence and the common tangent line of the two surfaces, so $P \in S$ and $P^* \in S^*$. Suppose that e_3, e_3^* are the normal vectors at points P and P^* to S and S^*, respectively. Finally, let τ be the angle between e_3 and e_3^*, so $\langle e_3, e_3^* \rangle = \cos\tau$, and let l be the distance between P and P^*. The following result will be invoked when required.

Theorem 1.1 (Bäcklund's Theorem) *Suppose that S and S^* are two focal surfaces of a pseudo-spherical congruence in $\mathbb{R}^3$, the distance between the corresponding points is constant and denoted l. The angle between the corresponding normals is a constant τ. Then these two focal surfaces S and S^* have the same negative constant Gaussian curvature*

$$K = -\frac{\sin^2\tau}{l^2}. \tag{1.2}$$

Thus, from any solution of the sine-Gordon or deformed sine-Gordon equation, a corresponding surface of negative constant curvature can be obtained. It is the latter case that is elucidated here.

On the other hand, from the Bäcklund theorem, it is known that two focal surfaces of a pseudospherical congruence are surfaces with the same negative constant curvature. These two focal surfaces will correspond to two solutions of the deformed sine-Gordon equation to appear. It will be seen that a relation can be established between the two solutions from the Bäcklund theorem, or equivalently, from the correspondence between two focal surfaces of a pseudo-spherical line congruence. This will be enough to give a Bäcklund transformation for this new deformed sine-Gordon equation.

2 Development of the Equation and Bäcklund Transformation.

Suppose S and S^* are two focal surfaces with arbitrary constant negative curvature K such that $\{P, e_1, e_2, e_3\}$ is a frame corresponding to coordinates of surface S with

$$\omega_1 = \cos\frac{\alpha}{2}\,du \qquad \omega_2 = \sin\frac{\alpha}{2}\,dv,$$
$$\omega_{13} = \sin\frac{\alpha}{2}\,du \quad \omega_{23} = -\cos\frac{\alpha}{2}\,du, \tag{2.1}$$

$$\omega_{12} = \frac{1}{2}(\alpha_v\,du + \alpha_u\,du) = -\omega_{21}.$$

These forms completely specify the set dr, de_1, de_2, de_3 in the fundamental equations given that $\omega_{ij} + \omega_{ji} = 0$.

Suppose

$$x^* = x + le = x + l(\cos\vartheta e_1 + \sin\vartheta e_2), \tag{2.2}$$

form a pseudo-spherical line congruence and ϑ is to be specified. In (2.2), x and x^* correspond to the surfaces S and S^*, l is the distance between the corresponding points P and P^* on the surfaces S and S^*, e is in the direction of PP^* and ϑ is the angle between e and e_1. Suppose S corresponds to a solution α of the deformed sine-Gordon equation to be obtained and α' a second solution. The fundamental equations for S are given by

$$dx = \omega_1 e_1 + \omega_2 e_2, \quad de_1 = \omega_{12} e_2 + \omega_{13} e_3, \quad de_2 = \omega_{21} e_1 + \omega_{23} e_3, \tag{2.3}$$

$$de_3 = \omega_{31} e_1 + \omega_{32} e_2, \quad \omega_3 = 0.$$

The fundamental equations for S^* are the same as (2.3), but with star on each quantity. By exterior differentiation of (2.2), it is found that

$$dx^* = dx + l(\cos\vartheta\,de_1 + \sin\vartheta\,de_2) + l(-\sin\vartheta\,e_1 + \cos\vartheta\,e_2)\,d\vartheta. \tag{2.4}$$

Using (2.1) in (2.3) and then substituting this into (2.4), there results,

$$dx^* = [\cos\frac{\alpha}{2}\,du - l\sin\vartheta\,d\vartheta - \frac{1}{2}l\sin\vartheta(\alpha_v\,du + \alpha_u\,dv)]e_1$$

$$+[\sin\frac{\alpha}{2}\,dv+\frac{1}{2}l\cos\vartheta(\alpha_v\,du+\alpha_u\,dv)+l\cos\vartheta\,d\vartheta]e_2+[l\sin\frac{\alpha}{2}\cos\vartheta\,du-l\cos\frac{\alpha}{2}\sin\vartheta\,dv]e_3. \tag{2.5}$$

Due to the fact that e_3^* has to be perpendicular to e_1 with respect to $\langle\,,\rangle$ and have a constant angle τ with respect to e_3, the unit normal of S^* at P^* takes the form

$$e_3^* = \sin\tau\sin\vartheta\,e_1 - \sin\tau\cos\vartheta\,e_2 + \cos\vartheta\,e_3. \tag{2.6}$$

Since e_3^* is the normal vector of S^*, with respect to the usual metric on $\mathbb{R}^3$

$$\langle dx^*, e_3^*\rangle = 0. \tag{2.7}$$

Calculating the left-hand side of (2.7) and simplifying, the following result is obtained

$$l\sin\tau\,d\vartheta + \frac{1}{2}l\sin\tau(\alpha_v\,du + \alpha_u\,dv)$$

$$-\sin\tau(\cos\frac{\alpha}{2}\sin\theta\,du-\sin\frac{\alpha}{2}\cos\vartheta\,dv)-l\cos\tau(\sin\frac{\alpha}{2}\cos\vartheta\,du-\cos\frac{\alpha}{2}\sin\vartheta\,dv) = 0. \tag{2.8}$$

Now ϑ is specified by considering the case in which

$$\vartheta = \frac{\alpha'}{2},$$

and since the orthogonality condition holds and du, dv are independent differentials, the coefficients in (2.8) can be equated to zero giving

$$\begin{aligned} \tfrac{1}{2}l\sin\tau\,(\alpha'_u + \alpha_v) &= \sin\tau\cos(\frac{\alpha}{2})\sin(\frac{\alpha'}{2}) + l\cos\tau\sin(\frac{\alpha}{2})\cos(\frac{\alpha'}{2}), \\ \tfrac{1}{2}l\sin\tau(\alpha'_v + \alpha_u) &= -\sin\tau\sin(\frac{\alpha}{2})\cos(\frac{\alpha'}{2}) - l\cos\tau\cos(\frac{\alpha}{2})\sin(\frac{\alpha'}{2}). \end{aligned} \tag{2.9}$$

No restrictions have been placed on the value of K up to this point. To give system (2.9) in another form, let us introduce a set of new variables σ, η defined to be

$$\sigma = \frac{1}{2}(u+v), \qquad \eta = \frac{1}{2}(u-v). \tag{2.10}$$

In terms of the variables (2.10), upon adding and subtracting the pair of equations in (2.9) and using standard trigonometric identities , they simplify to

$$\begin{gathered}\tfrac{1}{2} l \sin\tau (\alpha+\alpha')_{\sigma} = \sin\tau \sin(\frac{\alpha'-\alpha}{2}) + l \cos\tau \sin(\frac{\alpha-\alpha'}{2}), \\ \tfrac{1}{2} l \sin\tau (\alpha'-\alpha)_{\eta} = \sin\tau \sin(\frac{\alpha'+\alpha}{2}) + l \cos\tau \sin(\frac{\alpha+\alpha'}{2}).\end{gathered} \tag{2.11}$$

Introducing constants C_1 and C_2 to denote the pair of constants

$$C_1 = \frac{\sin\tau - l\cos\tau}{l\sin\tau}, \qquad C_2 = \frac{\sin\tau + l\cos\tau}{l\sin\tau}, \tag{2.12}$$

it is clear that (2.11) can be expressed in the equivalent form

$$(\alpha'+\alpha)_{\sigma} = 2C_1 \sin(\frac{\alpha'-\alpha}{2}), \qquad (\alpha'-\alpha)_{\eta} = 2C_2 \sin(\frac{\alpha'+\alpha}{2}). \tag{2.13}$$

System (2.13) will be compatible provided that the quantities α and α' satisfy a specific nonlinear equation. To obtain this equation, the compatibility condition for system (2.13) must be worked out. Differentiating, we obtain

$$\begin{gathered}(\alpha+\alpha')_{\sigma\eta} = 2C_1C_2 \cos(\frac{\alpha'-\alpha}{2}) \sin(\frac{\alpha'+\alpha}{2}), \\ (\alpha'-\alpha)_{\eta\sigma} = 2C_1C_2 \cos(\frac{\alpha'+\alpha}{2}) \sin(\frac{\alpha'-\alpha}{2}).\end{gathered} \tag{2.14}$$

Adding and subtracting the two in (2.14) and invoking trigonometric identities, it is found that both α and α' satisfy an identical deformed sine-Gordon equation, namely,

$$\psi_{\eta\sigma} = C_1C_2 \, \sin(\psi), \qquad \psi = \alpha, \alpha'. \tag{2.15}$$

Equation (2.15) can be expressed in terms of the u, v variables as follows

$$\psi_{uu} - \psi_{vv} = \frac{1}{2} C_1 C_2 \, \sin(\psi). \tag{2.16}$$

It should be remarked that, based on (2.12), the combination $C_1 C_2$ is not in general related in a straightforward way to K. The case in which $\sin \tau / l = 1$ can be considered separately. This corresponds to the case in which $K = -1$ so that

$$C_1 = \frac{1 - \cos \tau}{\sin \tau}, \qquad C_2 = \frac{1 + \cos \tau}{\sin \tau}.$$

In this case, it is easy to determine that

$$C_1 C_2 = \frac{1 - \cos^2 \tau}{\sin^2 \tau} = 1. \tag{2.17}$$

Therefore, corresponding to the case $K = -1$, upon setting $\beta = C_1$ and using (2.17) to get C_2, it is useful to note that system (2.13) assumes the usual form,

$$(\alpha' + \alpha)_\sigma = 2\beta \sin(\frac{\alpha' - \alpha}{2}), \qquad (\alpha' - \alpha)_\eta = \frac{2}{\beta} \sin(\frac{\alpha' + \alpha}{2}). \tag{2.18}$$

Let us make a summary of what has been done up to now. It has been seen Bäcklund's theorem has the following implications. Suppose S is a surface in $\mathbb{R}^3$ with negative, constant Gaussian curvature (1.2) such that $l > 0$ and $\tau \neq n\pi$ are constants. Let $e_0 \in T_{P_0} M$ be a unit vector which is not in the principle direction. Then there exists a unique surface S^* and a pseudo-spherical line congruence $\{PP^*\}$ where $P \in S$ and $P^* \in S^*$ satisfy $PP_0^* = l e_0$, and τ is the angle between the normal direction of S at P and S^* at P^*. The content of the new results is the next result.

Theorem 2.1 *A surface of arbitrary constant negative curvature (1.2) is determined by any nontrivial solution to (2.15)-(2.16) combined with the fundamental surface equations (2.3).*

3 Calculation of Solutions and Formulation of the Theorem of Permutability.

The system described by (2.13) is in fact a Bäcklund transformation for the equation (2.16). Given a particular solution to (2.16), it will be shown that (2.13) can be used to obtain a new solution to (2.16). Since $\alpha = 0$ is a solution to (2.15)-(2.16) for any C_1, C_2, substituting into (2.13) with $\alpha' = \alpha_1$, we have

$$\partial_\sigma \alpha_1 = 2C_1 \sin(\frac{\alpha_1}{2}), \qquad \partial_\eta \alpha_1 = 2C_2 \sin(\frac{\alpha_1}{2}). \tag{3.1}$$

Introduce two new variables s, t which are defined such that $s = C_1 \sigma$ and $t = C_2 \eta$ so that system (3.1) takes the form

$$\frac{\partial \alpha_1}{\partial s} = 2 \sin \frac{\alpha_1}{2}, \qquad \frac{\partial \alpha_1}{\partial t} = 2 \sin \frac{\alpha_1}{2}. \tag{3.2}$$

Since the derivatives in (3.2) are the same, it follows that α_1 must have the form $\alpha_1 = \alpha_1(s+t)$. To determine the form of the new solution α_1 corresponding to $\alpha = 0$ explicitly, let us write the first equation in (3.2) as

$$\frac{\partial \alpha_1}{\partial s} = 2 \sin(\frac{\alpha_1}{2}) = 4 \sin \frac{\alpha_1}{4} \cos \frac{\alpha_1}{4} = 4 \tan \frac{\alpha_1}{4} \cos \frac{\alpha_1}{4}. \tag{3.3}$$

This equation is equivalent to the form,

$$\frac{\partial}{\partial s} \tan \frac{\alpha_1}{4} = \tan \frac{\alpha_1}{4}. \tag{3.4}$$

In this form, the equation may be easily integrated with the help of (3.2) to give the solution

$$\tan \frac{\alpha_1}{4} = C \, \exp(s+t), \tag{3.5}$$

where C is an arbitrary real constant. It is now straightforward to transform back to the (u, v) variables from the (s, t) variables to yield

$$\tan \frac{\alpha_1}{4} = C \exp[C_1 \sigma + C_2 \eta] = C \exp[\frac{1}{2} C_1 (u+v) + \frac{1}{2} C_2 (u-v)]. \tag{3.6}$$

Therefore a new solution to deformed sine-Gordon equation (2.15) has been found starting with the $\alpha = 0$ solution applying (2.13) and integrating. Let us summarize it in the form,

$$\alpha_1 = 4\tan^{-1}\{C\exp[\frac{1}{2}(C_1 + C_2)u + \frac{1}{2}(C_1 - C_2)v]\}.$$

Other solutions to (2.15) can be constructed along similar lines.

According to the theorem of permutability, the application of two successive Bäcklund transformations commutes. To express it more quantitatively, if two successive Bäcklund transformations with distinct parameters λ_1 and λ_2 map a given solution α_0 through intermediate solutions to either α_{12} or α_{21}, the order in which this is done is irrelevant and in fact $\alpha_{12} = \alpha_{21}$. If the intermediate solutions are denoted α_1 and α_2, then making use of the η equation in (2.13) and identifying the Bäcklund parameter as the constant which appears on the right, the scheme described can be expressed in the form

$$(\alpha_1 - \alpha_0)_\eta = 2\lambda_1 \sin(\frac{\alpha_1 + \alpha_0}{2}),$$

$$(\alpha_{12} - \alpha_1)_\eta = 2\lambda_2 \sin(\frac{\alpha_{12} + \alpha_1}{2}),$$

$$(\alpha_2 - \alpha_0)_\eta = 2\lambda_2 \sin(\frac{\alpha_2 + \alpha_0}{2}),$$

$$(\alpha_{12} - \alpha_2)_\eta = 2\lambda_1 \sin(\frac{\alpha_{12} + \alpha_2}{2}).$$

In fact, all the derivative terms can be eliminated from these equations. Subtracting the first two and the last two pairwise, and then subtracting these two resulting equations produces the result,

$$\lambda_2(\sin(\frac{\alpha_{12} + \alpha_1}{2}) - \sin(\frac{\alpha_2 + \alpha_0}{2})) + \lambda_1(\sin(\frac{\alpha_1 + \alpha_0}{2}) - \sin(\frac{\alpha_{12} + \alpha_2}{2})) = 0. \tag{3.7}$$

By making use of standard trigonometric identities, it is possible to render this in the following concise form,

$$(\lambda_2 - \lambda_1)\tan(\frac{\alpha_{12} - \alpha_0}{4}) = (\lambda_1 + \lambda_2)\tan(\frac{\alpha_2 - \alpha_0}{4}). \tag{3.8}$$

The usual result for the sine-Gordon equation is obtained. The theorem of permutability allows the construction algebraically of a second order solution, and the procedure can be carried out order by order.

To conclude, it has been seen here that the sine-Gordon equation has been generalized to accommodate cases of arbitrary Gaussian curvature, and a Bäcklund transformation has been calculated as well as applying it to generate a solution. Further solutions can be produced from it using the theorem of permutability.

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