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Further result of Schwick normal criterion for families of holomorphic functions

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Abstract

In the paper, we prove a result: Let $k(\geq 2)$ be a positive integer and let $\mathcal{F}$ be a family of functions holomorphic in a domain $D \subseteq \mathbf{C}$ and all of whose zeros have multiplicity at least k. Suppose that $f(z)$ and $f^{(k)}(z)$ share zero IM in D for all $f \in \mathcal{F}$. Then $\{\frac{f'}{f} : f \in \mathcal{F}\}$ is normal in D. Our result extend the Schwick [10] normal criteria in which suppose that $f(z)$ and $f^{(k)}(z)$ have no zeros in D for all $f \in \mathcal{F}$.

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1 Introduction and Main Result

Let f be nonconstant meromorphic (entire) function in the whole plane. It is assumed that the reader is familiar with the standard notations and the basic results of Nevanlinna's value-distribution theory(see Hayman [6] or Schiff [9]).

Let D be a domain in complex plain $\mathbf{C}$ and $\mathcal{F}$ a family of meromorphic functions defined in D. $\mathcal{F}$ is called to be normal in D if each sequence $\{f_n\} \subset \mathcal{F}$ has a subsequence $\{f_{n_k}\}$ which converges spherically locally uniformly in D to a meromorphic function or ∞(see Schiff [9]).

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in a domain $D \subseteq \mathbf{C}$, and let a be a finite complex value. We say that f and g share a CM(or IM) provided that $f-a$ and $g-a$ have the same zeros counting(or ignoring) multiplicity in D.

The following theorem was conjectured by Hayman [5] in 1959 and proved by Frank [4] in 1976 for $k \geq 3$ and by Langley [7] in 1993 for $k = 2$.

Theorem 1.1 *Let $f(z)$ be a nonconstant meromorphic function in the whole complex plane $\mathbf{C}$ and $k \geq 2$. If f and $f^{(k)}$ have no zeros, then $f(z)$ has the form $f(z) = e^{az+b}$ or $f(z) = (az+b)^{-m}$, where $a, b \in \mathbf{C}, a \neq 0, m \in \mathbf{N}$.*

In the case where f is entire the result was proved by Hayman [5] himself for $k = 2$ and by Clunie [3] in the general case. In this case $\frac{f'}{f}$ is constant. Influenced from Bloch's principle [1]), that is, there is a normality criterion corresponding to every Liouville-Picard type theorem. The following result of Schwick [10] can be considered as the normal families analogue arising according to Bloch's Principle from Theorem 1.1 restricted to entire functions.

Theorem 1.2 *Let $k(\geq 2)$ be a positive integer and let $\mathcal{F}$ be a family of functions holomorphic in a domain $D \subseteq \mathbf{C}$. If f and $f^{(k)}$ have no zeros in D for all $f \in \mathcal{F}$, then $\{\frac{f'}{f} : f \in \mathcal{F}$ is normal in D.*

In this paper, we extend Theorem 1.2.

Theorem 1.3 (Main Result) *Let $k(\geq 2)$ be a positive integer and let $\mathcal{F}$ be a family of functions holomorphic in a domain $D \subseteq \mathbf{C}$ and all of whose zeros have multiplicity at least k. Suppose that $f(z)$ and $f^{(k)}(z)$ share zero IM in D for all $f \in \mathcal{F}$. Then $\{\frac{f'}{f} : f \in \mathcal{F}\}$ is normal in D.*

The following Example 1.1 illustrates that Theorem 1.3 is more efficient than Theorem 1.2.

Example 1.1 *Take $k(\geq 2)$ be a fixed positive integer, $f_n(z) = z^n, n \geq k$, $D = \{|z| < 1\}$. Then $f_n(z)$ and $f_n^{(k)}(z)$ have the only one distinct zero $z = 0$. So $f_n(z)$ and $f_n^{(k)}(z)$ share 0 IM in D and $\{\frac{f_n'(z)}{f_n(z)}\}$ is normal in D by Theorem 1.1*

In 2003, Bergweiler and Langley [2] extended the Schwick's theorem to families of meromorphic functions and obtained the following theorem.

Theorem 1.4 *Let $k(\geq 2)$ be a positive integer and let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subseteq \mathbf{C}$. If f and $f^{(k)}$ have no zeros in D for all $f \in \mathcal{F}$, then $\{\frac{f'}{f} : f \in \mathcal{F}\}$ is normal in D.*

It is natural to pose the following conjecture.

Conjecture 1.2 Let $k(\geq 2)$ be a positive integer and let $\mathcal{F}$ be a family of functions meromorphic in a domain $D \subseteq \mathbf{C}$ and all of whose zeros have multiplicity at least k. Suppose that $f(z)$ and $f^{(k)}(z)$ share zero IM in D for all $f \in \mathcal{F}$. Then $\{\frac{f'}{f} : f \in \mathcal{F}\}$ is normal in D.

2 Preliminary Lemmas

In order to prove our result, we need the following lemmas. Lemma 2.1 is an extending result of Zalcman[8] concerning normal families.

Lemma 2.1 [8] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disc, all of whose zeros have multiplicity at least k, and there exist $A \geq 1$ such that $f(z) = 0$ implies $|f^{(k)}(z)| \leq A$ for each $f(z) \in \mathcal{F}$. If $\mathcal{F}$ is not normal on the unit disc, then for any $0 \leq \alpha \leq k$ there exist*

a) a number $0 < r < 1$;
b) points z_n with $|z_n| < r$;
c) functions $f_n \in \mathcal{F}$;
d) positive numbers $\rho_n \to 0$

such that $g_n(\zeta) := \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta)$ converges locally uniformly to a nonconstant meromorphic function $g(\zeta)$ whose zeros have multiplicity at least k and order is at most 2, and $g^\sharp(\zeta) \leq g^\sharp(0) = kA + 1$.

Remark $g(\zeta)$ is a nonconstant entire function if $\mathcal{F}$ is a family of holomorphic functions on the unit disc in Lemma 2.1.

In order to state the following lemmas [2] and the proof of Theorem 1.1, we define differential operators Ψ_k for $k \in \mathbf{N}$ by

$$\Psi_1(y), \Psi_{k+1}(y) = y\Psi_k(y) + \frac{d}{dz}(\Psi_k(y)). \tag{1}$$

Lemma 2.2 *Let $f(z)$ be meromorphic in a domain D and let $F(z) := \frac{f'}{f}$. Then for each $k \in \mathbf{N}$ we have $\Psi_k(F) = \frac{f^{(k)}}{f}$.*

Lemma 2.3 *Let $k \geq 2$ be an integer, and let $F(z)$ be nonconstant and meromorphic in the whole complex plane $\mathbf{C}$ and satisfy the following conditions:*

(1) $\Psi_k(F)$ has no zeros;

(2) if z_0 is a simple pole of F then $\mathrm{Res}(F, z_0) \notin \{1, 2, \cdots, k-1\}$;

(3) for $k = 2$, there exists $\delta > 0$ such that if z_0 is a simple pole of F then $|\mathrm{Res}(F, z_0)| \geq \delta$.

Then F has the form

$$F(z) = \frac{(k-1)z + a}{z^2 + bz + c} \tag{2}$$

or

$$F(z) = \frac{1}{az + b}. \tag{3}$$

Here $a, b, c \in \mathbf{C}$, with $a \neq 0$ in (3).

Lemma 2.4 *Let $k \geq 2$ be an integer, and let y be meromorphic in a domain D, such that if z_0 is a simple pole of y then* $\mathrm{Res}(y, z_0) \notin \{1, 2, \cdots, k-1\}$. *Let $k \in \mathbf{N}$ with $n \leq k$. If y has a pole at z_0 of multiplicity m then $\Psi_n(y)$ has a pole at z_0 of multiplicity nm.*

3 Proof of Main Result

Without loss of generality, we assume that $D = \{z \in \mathbf{C}, |z| < 1\}$. Suppose that $\{\frac{f'}{f} : f \in \mathcal{F}\}$ is not normal in D. Without loss of generality, we assume that $\{\frac{f'}{f} : f \in \mathcal{F}\}$ is not normal at $z = 0$.

Claim 1. $\frac{f^{(k)}}{f}$ have no zeros and $\mathrm{Res}(\frac{f'}{f}, z_0) \geq k$ for any pole z_0 of $\frac{f'}{f}$.

By the hypothesis of Theorem 1.3, we know that $f^{(k)}$ and f sharing zero IM implies that $\frac{f^{(k)}}{f}$ have no zeros. Since f is holomorphic in D and whose zeros have multiplicity at least k, we see that any pole z_0 of $\frac{f'}{f}$ must be simple and is a zero of f and $\mathrm{Res}(\frac{f'}{f}, z_0) \geq k$ holds.

Furthermore, $\mathrm{Res}(\frac{f'}{f}, z_0) \notin \{1, 2, \cdots, k-1\}$. Thus take $\delta \in (0, k]$, we have that $|\mathrm{Res}(\frac{f'}{f}, z_0)| \geq \delta$ for any zero z_0 of $f \in \mathcal{F}$. Applying Lemma 2.1 to the family of all functions $\frac{1}{F}$ with $F := \frac{f'}{f} \in \{\frac{f'}{f} : f \in \mathcal{F}\}$,we obtain that there exist a sequence of points $z_n \to 0$, $f_n \in \mathcal{F}$ and $\rho_n \to 0^+$ such that as $n \to \infty$

$$g_n(\zeta) := \rho_n F_n(z_n + \rho_n \zeta) \to g(\zeta)$$

uniformly on any compact subset of $\mathbf{C}$, where $g(\zeta)$ is a non-constant meromorphic function such that $g^\sharp(z) \leq g^\sharp(0) = 1 + \frac{1}{\delta}$ for all $z \in \mathbf{C}$.

Let z_0 is a simple pole of g. Then, By Hurwitz theorem, if n is large enough, g_n has a simple pole a_n with $a_n \to z_0$. Since $z_n + \rho_n a_n$ is a simple pole of F_n with $\mathrm{Res}(F_n, z_n + \rho_n a_n) = \mathrm{Res}(g_n, a_n)$ we deduce from Claim 1 that $\mathrm{Res}(g_n, a_n) \geq k \geq \delta$. This implies that

Claim 2. $\mathrm{Res}(g, z_0) \geq k \geq \delta$. In particular, every pole of g is a pole of $\Psi_k(g)$, by Lemma 2.4.

Note that $\Psi_k(g_n(\zeta)) = \rho_n^k \Psi_k(F_n(z_n + \rho_n \zeta))$, we get by Lemma 2.2 and Claim 1 that $\Psi_k(g_n)$ has no zeros. Set S be the set of poles of g, then $\Psi_k(g_n) \to \Psi_k(g)$

locally uniformly on $\mathbf{C}\setminus S$, and either $\Psi_k(g)$ has no zeros or $\Psi_k(g) \equiv 0$ on $\mathbf{C}\setminus S$ by Hurwitz theorem. In the former case we know that $\Psi_k(g)$ has no zeros and that $\Psi_k(g_n) \to \Psi_k(g)$ on the whole plane $\mathbf{C}$ by the maximum principle applied to $\frac{1}{\Psi_k(g_n)}$ and $\frac{1}{\Psi_k(g)}$.

Case 1. $\Psi_k(g)$ has no zeros.

By Lemma 2.3, we deduce that g has the form (2) or (3).

Suppose that g has the form (2) but is not of the form (3). Then g has two poles, counting multiplicities, and

$$\sum_{z_0 \in g^{-1}(\{\infty\})} \mathrm{Res}(g, z_0) = k - 1, \tag{4}$$

by the residue theorem.

On the other hand, by Claim 2, we infer that

$$\sum_{z_0 \in g^{-1}(\{\infty\})} \mathrm{Res}(g, z_0) \geq k.$$

This contradicts with (4).

Suppose that g has the form (3). Then $\frac{1}{|a|} = |\mathrm{Res}(g, -\frac{b}{a})| \geq \delta$ so that $|a| \leq \frac{1}{\delta}$. On the other hand, $|a| \geq \frac{|a|}{1+|b|^2} = g^\sharp(0) = \frac{1}{\delta}$. This is impossible.

Case 2. $\Psi_k(g) \equiv 0$.

By Claim 2, we have from Lemma 2.2 that g has no poles. Thus g is entire, and so is the function f defined by $f(z) = \exp(\int_0^z g(\zeta)d\zeta)$. Hence $g = \frac{f'}{f}$, $\frac{f^{(k)}}{f} = \Psi_k(g) \equiv 0$ by Lemma 2.2. Therefore f is a polynomial. This implies that f is constant, and then $g \equiv 0$, a contradiction.

The proof of Theorem 1.3 is complete.

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