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A Survey of an Analog of Analytic Feynman Integrals on a Function Space

Myung Jae Kim

Department of Mathematics
Kyonggi University, Suwon 443-760, Korea
mjkim@kyonggi.ac.kr

Abstract

Let $C^r[0,t]$ denote an analogue of the r-dimensional Wiener space. In the present paper, we introduce analogues of the analytic Wiener and Feynman integrals for several types of functions, in particular, the functionals of the forms

$$\exp\left\{\int_0^t \theta(s,x(s))d\eta(s)\right\}\psi(x(t)) \text{ and } \sum_{j=1}^{r}\int_0^t (x_j(s))^{m_j}ds$$

for $x=(x_1,\cdots,x_r)\in C^r[0,t]$, where η is a complex Borel measure on $[0,t]$, and $\theta(s,\cdot)$ and ψ are the Fourier-Stieltjes transforms of the complex Borel measures on the r-dimensional Euclidean space $\mathbb{R}^r$.

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1 Introduction and Preliminaries

Let $C_0[0,t]$ denote the classical Wiener space which is the space of all paths x on $[0,t]$ with $x(0)=0$. The analytic Feynman integrals of various type of functionals on $C_0[0,t]$ were evaluated in [1, 2, 7, 10]. Further works were studied on $C[0,t]$, the space of the real-valued continuous functions on the interval $[0,t]$ [3, 4, 5, 6, 9]. For more details, let $0=t_0<t_1<\cdots<t_n<t_{n+1}=t$ be a partition of the interval $[0,t]$. Im and Ryu [6, 9] introduced a probability measure w_φ on $(C[0,t],\mathcal{B}(C[0,t]))$ where $\mathcal{B}(C[0,t])$ denotes the Borel σ-algebra on $C[0,t]$ and φ is a probability measure on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$. On the

space $C[0,t]$, Cho [3] evaluated the conditional analytic Feynman w_φ-integrals of the functions of the form

$$F_m(x) = \int_0^t (x(s))^m ds, \quad m = 1, 2, \cdots$$

with the conditioning functions $X_n : C[0,t] \to \mathbb{R}^{n+1}$ given by

$$X_n(x) = (x(t_0), x(t_1), \cdots, x(t_n))$$

and $X_{n+1} : C[0,t] \to \mathbb{R}^{n+2}$ given by

$$X_{n+1}(x) = (x(t_0), x(t_1), \cdots, x(t_n), x(t_{n+1})).$$

He and his co-workers [5] established the same kinds of the conditional Feynman w_φ-integrals for the cylinder type functionals on space $C[0,t]$. On the product space $C^r[0,t]$ of $C[0,t]$, he [4] also derived the evaluation formulas of similar kinds of the conditional Feynman w_φ^r-integrals for the functionals having the form

$$\exp\left\{\int_0^t \theta(s, x(s)) d\eta(s)\right\}\psi(x(t)), \quad x \in C^r[0,t],$$

where η is a complex Borel measure on $[0,t]$, and $\theta(s,\cdot)$ and ψ are the Fourier-Stieltjes transforms of the complex Borel measures on the r-dimensional Euclidean space.

In present paper, we evaluate analogues of the analytic Wiener and Feynman integrals for the cylinder type functions on $C[0,t]$ and the functionals of forms

$$\exp\left\{\int_0^t \theta(s, x(s)) d\eta(s)\right\}\psi(x(t)) \text{ and } \sum_{j=1}^r \int_0^t (x_j(s))^{m_j} ds$$

for $x = (x_1, \cdots, x_r) \in C^r[0,t]$. On the space $C^r[0,t]$, we also establish the same kinds of the evaluation formulas for the functionals in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ [2].

Throughout this paper, let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of the complex numbers and the complex numbers with positive real parts, respectively. We also let m_L be the Lebesgue measure on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$.

Now, we begin with introducing the probability measure w_φ on $(C[0,t], \mathcal{B}(C[0,t]))$. For a positive real t, let $C = C[0,t]$ be the space of all real-valued continuous functions on the closed interval $[0,t]$ with the supremum norm. For $\vec{t} = (t_0, t_1, \cdots, t_n)$ with $0 = t_0 < t_1 < \cdots < t_n \le t$, let $J_{\vec{t}} : C[0,t] \to \mathbb{R}^{n+1}$ be the function given by

$$J_{\vec{t}}(x) = (x(t_0), x(t_1), \cdots, x(t_n)).$$

For $B_j\,(j=0,1,\cdots,n)$ in $\mathcal{B}(\mathbb{R})$, the subset $J_{\vec{t}}^{-1}(\prod_{j=0}^{n} B_j)$ of $C[0,t]$ is called an interval and let $\mathcal{I}$ be the set of all such intervals. For a probability measure φ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, let

$$
\begin{aligned}
& m_\varphi\biggl(J_{\vec{t}}^{-1}\biggl(\prod_{j=0}^{n} B_j\biggr)\biggr) \\
= \;& \biggl[\prod_{j=1}^{n}\frac{1}{2\pi(t_j-t_{j-1})}\biggr]^{\frac{1}{2}}\int_{B_0}\int_{\prod_{j=1}^{n}B_j}\exp\biggl\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\biggr\} \\
& dm_L^n(u_1,\cdots,u_n)d\varphi(u_0).
\end{aligned}
$$

$\mathcal{B}(C[0,t])$, the Borel σ-algebra on $C[0,t]$ coincides with the smallest σ-algebra generated by $\mathcal{I}$ and there exists a unique probability measure w_φ on $(C[0,t],\mathcal{B}(C[0,t]))$ such that $w_\varphi(I)=m_\varphi(I)$ for all I in $\mathcal{I}$. This measure w_φ is called an analogue of the Wiener measure associated with the probability measure φ [6, 9]. Let r be a positive integer and $C^r=C^r[0,t]$ be the product space of $C[0,t]$ with the product measure w_φ^r. Since $C[0,t]$ is a separable Banach space, we have $\mathcal{B}(C^r[0,t])=\prod_{j=1}^{r}\mathcal{B}(C[0,t])$. This probability measure space $(C^r[0,t],\mathcal{B}(C^r[0,t]),w_\varphi^r)$ is called an analogue of the r-dimensional Wiener space.

Lemma 1.1 ([6, Lemma 2.1]) *If $f:\mathbb{R}^{n+1}\to\mathbb{C}$ is a Borel measurable function, then*

$$
\begin{aligned}
& \int_C f(x(t_0),x(t_1),\cdots,x(t_n))dw_\varphi(x) \\
\overset{*}{=}\;& \biggl[\prod_{j=1}^{n}\frac{1}{2\pi(t_j-t_{j-1})}\biggr]^{\frac{1}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} f(u_0,u_1,\cdots,u_n)\exp\biggl\{-\frac{1}{2}\sum_{j=1}^{n}\frac{(u_j-u_{j-1})^2}{t_j-t_{j-1}}\biggr\} \\
& dm_L^n(u_1,\cdots,u_n)d\varphi(u_0)
\end{aligned}
$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

Let $\{e_k : k=1,2,\cdots\}$ be a complete orthonormal subset of $L_2[0,t]$ such that each e_k is of bounded variation. For f in $L_2[0,t]$ and x in $C[0,t]$, we let

$$(f,x)=\lim_{n\to\infty}\sum_{k=1}^{n}\langle f,e_k\rangle\int_0^t e_k(s)dx(s)$$

if the limit exists. Here $\langle\cdot,\cdot\rangle$ denotes the inner product over $L_2[0,t]$. (f,x) is called the Paley-Wiener-Zygmund integral of f according to x. Note that $\langle\cdot,\cdot\rangle$ also denotes the inner product over Euclidean space unless otherwise specified.

Applying [6, Theorem 3.5], we can easily prove the following theorem.

Theorem 1.2 *Let $\{h_1, h_2, \cdots, h_n\}$ be an orthonormal system of $L_2[0,t]$. For $i = 1, 2, \cdots, n$, let $Z_i(x) = (h_i, x)$ on $C[0,t]$. Then $Z_1, Z_2, \cdots, Z_n$ are independent and each Z_i has the standard normal distribution. Moreover, if $f : \mathbb{R}^n \to \mathbb{R}$ is Borel measurable, then*

$$\begin{aligned}&\int_C f(Z_1(x), Z_2(x), \cdots, Z_n(x))dw_\varphi(x)\\ \overset{*}{=}\ &\left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} f(u_1, u_2, \cdots, u_n) \exp\left\{-\frac{1}{2}\sum_{j=1}^n u_j^2\right\} dm_L^n(u_1, u_2, \cdots, u_n),\end{aligned}$$

where $\overset{*}{=}$ means that if either side exists then both sides exist and they are equal.

For a function $F : C^r[0,t] \to \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$. If $E[F^\lambda]$ has the analytic extension $J^*_\lambda(F)$ on $\mathbb{C}_+$, then we call $J^*_\lambda(F)$ the analytic Wiener w^r_φ-integral of F over $C^r[0,t]$ with the parameter λ and it is denoted by

$$E^{anw_\lambda}[F] = J^*_\lambda(F).$$

Further, if for a nonzero real q, $E^{anw_\lambda}[F]$ has the limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the analytic Feynman w^r_φ-integral of F over $C^r[0,t]$ with the parameter q and denoted by

$$E^{anf_q}[F] = \lim_{\lambda \to -iq} E^{anw_\lambda}[F].$$

2 Evaluation Formulas for Feynman Integrals

In this section, we establish the evaluation formulas for the analytic Feynman w^r_φ-integrals of several kinds of functionals on the analogue of the r-dimensional Wiener space.

Theorem 2.1 *Let $m_1, \cdots, m_r$ be positive integers and suppose that*

$$\int_{\mathbb{R}} |u|^m d\varphi(u) < \infty,$$

where $m = \max\{m_1, \cdots, m_r\}$. For $x = (x_1, \cdots, x_r) \in C^r[0,t]$, let

$$F(x) = \sum_{j=1}^r \int_0^t (x_j(s))^{m_j} ds.$$

Then for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F]$ exists and it is given by

$$E^{anw_\lambda}[F] = \sum_{j=1}^r \lambda^{-\frac{m_j}{2}} \sum_{k=0}^{\left[\frac{m_j}{2}\right]} \frac{t^{k+1}}{k+1} \frac{m_j!}{2^k k!(m_j-2k)!} \int_{\mathbb{R}} v_j^{m_j-2k} d\varphi(v_j),$$

where $[\,\cdot\,]$ *denotes the greatest integer function. Furthermore, for a nonzero real* q*,* $E^{anf_q}[F]$ *exists and it is given by the right hand side of the above equality where* λ *is replaced by* $-iq$*.*

Proof. For $\lambda > 0$, we have

$$\begin{aligned} E[F^\lambda] &= \sum_{j=1}^{r}\int_{C^r}\int_0^t (\lambda^{-\frac{1}{2}}x_j(s))^{m_j}\,ds\,dw_\varphi^r(x) \\ &= \sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\int_0^t\Big(\frac{1}{2\pi s}\Big)^{\frac{r}{2}}\int_{\mathbb{R}^r}\int_{\mathbb{R}^r}u_j^{m_j}\exp\Big\{-\frac{1}{2s}\sum_{k=1}^{r}(u_k-v_k)^2\Big\} \\ &\quad dm_L^n(u_1,\cdots,u_r)\,d\varphi^r(v_1,\cdots,v_r)\,ds \end{aligned}$$

where the last equality follows from Lemma 1.1. By the binomial expansion we have

$$\begin{aligned} &E[F^\lambda] \\ &= \sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\int_0^t\Big(\frac{1}{2\pi s}\Big)^{\frac{1}{2}}\int_{\mathbb{R}}\int_{\mathbb{R}}(u_j+v_j)^{m_j}\exp\Big\{-\frac{1}{2s}u_j^2\Big\}dm_L(u_j)\,d\varphi(v_j)\,ds \\ &= \sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\int_0^t\int_{\mathbb{R}}\sum_{l=0}^{m_j}\binom{m_j}{l}v_j^{m_j-l}\Big(\frac{1}{2\pi s}\Big)^{\frac{1}{2}}\int_{\mathbb{R}}u_j^l\exp\Big\{-\frac{1}{2s}u_j^2\Big\}dm_L(u_j) \\ &\quad d\varphi(v_j)\,ds \\ &= \sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\int_0^t\int_{\mathbb{R}}\sum_{k=0}^{[\frac{m_j}{2}]}\binom{m_j}{2k}v_j^{m_j-2k}\Big(\frac{2}{\pi s}\Big)^{\frac{1}{2}}\int_0^\infty u_j^{2k}\exp\Big\{-\frac{1}{2s}u_j^2\Big\}du_j \\ &\quad d\varphi(v_j)\,ds, \end{aligned}$$

where $[\,\cdot\,]$ denotes the greatest integer function. Let $z_j = \frac{1}{2s}u_j^2$ for $j = 1,\cdots,r$. Then $u_j = (2sz_j)^{\frac{1}{2}}$ so that $\frac{du_j}{dz_j} = (\frac{s}{2z_j})^{\frac{1}{2}}$. By the change of variable theorem,

$$\begin{aligned} &E[F^\lambda] \\ &= \Big(\frac{1}{\pi}\Big)^{\frac{1}{2}}\sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\int_0^t\int_{\mathbb{R}}\sum_{k=0}^{[\frac{m_j}{2}]}\binom{m_j}{2k}v_j^{m_j-2k}(2s)^k\int_0^\infty z_j^{k-\frac{1}{2}}\exp\{-z_j\} \\ &\quad dz_j\,d\varphi(v_j)\,ds \\ &= \Big(\frac{1}{\pi}\Big)^{\frac{1}{2}}\sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\int_0^t\int_{\mathbb{R}}\sum_{k=0}^{[\frac{m_j}{2}]}\binom{m_j}{2k}v_j^{m_j-2k}(2s)^k\Gamma\Big(k+\frac{1}{2}\Big)d\varphi(v_j)\,ds \\ &= \Big(\frac{1}{\pi}\Big)^{\frac{1}{2}}\sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\int_0^t\int_{\mathbb{R}}\sum_{k=0}^{[\frac{m_j}{2}]}\binom{m_j}{2k}v_j^{m_j-2k}(2s)^k\Big(k-\frac{1}{2}\Big)\Big(k-\frac{3}{2}\Big)\cdots \end{aligned}$$

$$\frac{1}{2}\Gamma\left(\frac{1}{2}\right)d\varphi(v_j)ds$$

$$= \sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\int_0^t\int_{\mathbb{R}}\sum_{k=0}^{[\frac{m_j}{2}]}\frac{m_j!}{(m_j-2k)!(2k)!}v_j^{m_j-2k}s^k 1\cdot 3\cdot\cdots\cdot(2k-3)(2k$$
$$-1)d\varphi(v_j)ds$$

$$= \sum_{j=1}^{r}\lambda^{-\frac{m_j}{2}}\sum_{k=0}^{[\frac{m_j}{2}]}\frac{t^{k+1}}{k+1}\frac{m_j!}{2^k k!(m_j-2k)!}\int_{\mathbb{R}}v_j^{m_j-2k}d\varphi(v_j).$$

Now the theorem follows. □

Example 2.2 *Suppose that* $\int_{\mathbb{R}}|u|^2 d\varphi(u) < \infty$. *For* $x = (x_1, \cdots, x_r) \in C^r[0,t]$, *let*

$$F_l(x) = \sum_{j=1}^{r}\int_0^t (x_j(s))^l ds, \quad l = 1, 2.$$

For a nonzero real q, *we have by Theorem 2.1*

$$E^{anf_q}[F_1] = t\left(\frac{i}{q}\right)^{\frac{1}{2}}\sum_{j=1}^{r}\int_{\mathbb{R}}v_j d\varphi(v_j)$$

and

$$E^{anf_q}[F_2] = \frac{i}{q}\bigg(t\sum_{j=1}^{r}\int_{\mathbb{R}}u_j^2 d\varphi(v_j) + \frac{rt^2}{2}\bigg).$$

Let $\mathcal{M} = \mathcal{M}(L_2^r[0,t])$ be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2^r[0,t]$ and let $\mathcal{S}_{w_\varphi}^r$ be the space of all functions F which for $\sigma \in \mathcal{M}$ have the form

$$F(x) = \int_{L_2^r[0,t]}\exp\bigg\{i\sum_{j=1}^{r}(v_j, x_j)\bigg\}d\sigma(v_1, \cdots, v_r) \tag{1}$$

for w_φ^r-a.e. $x = (x_1, \cdots, x_r) \in C^r[0,t]$. Note that $\mathcal{S}_{w_\varphi}^r$ is a Banach algebra which is equivalent to $\mathcal{M}$ with the norm $\|F\| = \|\sigma\|$, the total variation of σ [6].

Now, we have the following theorem.

Theorem 2.3 *Let* $F \in \mathcal{S}_{w_\varphi}^r$ *be given by (1). Then for* $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[F]$ *exists and it is given by*

$$E^{anw_\lambda}[F] = \int_{L_2^r[0,t]}\exp\bigg\{-\frac{1}{2\lambda}\sum_{j=1}^{r}\|v_j\|^2\bigg\}d\sigma(\vec{v}),$$

where $\vec{v} = (v_1, \cdots, v_r)$*. Furthermore, for nonzero real* q*,* $E^{anf_q}[F]$ *exists and it is given by the right hand side of the above equality where* λ *is replaced by* $-iq$*.*

Proof. For $\lambda > 0$

$$\begin{aligned} E[F^\lambda] &= \int_{C^r}\int_{L_2^r[0,t]} \exp\Big\{ i\lambda^{-\frac{1}{2}} \sum_{j=1}^r (v_j, x_j) \Big\} d\sigma(\vec{v}) dw_\varphi^r(x) \\ &= \int_{L_2^r[0,t]} \prod_{j=1}^r \Big(\frac{1}{2\pi \|v_j\|^2} \Big)^{\frac{1}{2}} \int_{\mathbb{R}^r} \exp\Big\{ i\lambda^{-\frac{1}{2}} \sum_{j=1}^r u_j - \frac{1}{2} \sum_{j=1}^r \frac{u_j^2}{\|v_j\|^2} \Big\} \\ &\quad dm_L^n(u_1, \cdots, u_r) d\sigma(\vec{v}) \\ &= \int_{L_2^r[0,t]} \exp\Big\{ -\frac{1}{2\lambda} \sum_{j=1}^r \|v_j\|^2 \Big\} d\sigma(\vec{v}) \end{aligned}$$

where the last equality follows from the well-known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \Big(\frac{\pi}{a} \Big)^{\frac{1}{2}} \exp\Big\{ -\frac{b^2}{4a} \Big\} \tag{2}$$

for $a \in \mathbb{C}_+$ and any real b. Now, the theorem follows from the Morera's theorem and the dominated convergence theorem. □

Let $r = 1$ and $\{e_1, \cdots, e_l\}$ be an orthonormal subset of $L_2[0,t]$. For $1 \le p \le \infty$, let $\mathcal{A}_l^{(p)}$ be the set of cylinder type functions having the form

$$F_l(x) = f((e_1, x), \cdots, (e_l, x)) \tag{3}$$

for w_φ-a.e. $x \in C[0,t]$, where $f \in L_p(\mathbb{R}^l)$ is Borel measurable on $\mathbb{R}^l$. We now have the following theorem.

Theorem 2.4 *Let* $F_l \in \mathcal{A}_l^{(p)}$ $(1 \le p \le \infty)$ *be given by (3). Then for* $\lambda \in \mathbb{C}_+$*,* $E^{anw_\lambda}[F_l]$ *exists and it is given by*

$$E^{anw_\lambda}[F_l] = \Big(\frac{\lambda}{2\pi} \Big)^{\frac{l}{2}} \int_{\mathbb{R}^l} f(\vec{u}) \exp\Big\{ -\frac{\lambda}{2} \|\vec{u}\|^2 \Big\} dm_L^l(\vec{u}).$$

Furthermore, for nonzero real q*,* $E^{anf_q}[F_l]$ *exists if* $p = 1$ *and it is given by the right hand side the above equality where* λ *is replaced by* $-iq$*.*

Proof. For $\lambda > 0$, we have by Theorem 1.2 and the change of variable theorem

$$\begin{aligned} E[F_l^\lambda] &= \int_C f(\lambda^{-\frac{1}{2}}((e_1, x), \cdots, (e_l, x))) dw_\varphi(x) \\ &= \Big(\frac{1}{2\pi} \Big)^{\frac{l}{2}} \int_{\mathbb{R}^l} f(\lambda^{-\frac{1}{2}} \vec{u}) \exp\Big\{ -\frac{1}{2} \|\vec{u}\|^2 \Big\} dm_L^l(\vec{u}) \end{aligned}$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{l}{2}} \int_{\mathbb{R}^l} f(\vec{u}) \exp\left\{-\frac{\lambda}{2}\|\vec{u}\|^2\right\} dm_L(\vec{u}).$$

The theorem now follows from the Morera's theorem and the dominated convergence theorem. □

Let $\hat{\mathrm{M}}(\mathbb{R}^l)$ be the set of all functions ϕ on $\mathbb{R}^l$ defined by

$$\phi(\vec{u}) = \int_{\mathbb{R}^l} \exp\{i\langle\vec{u},\vec{z}\rangle\} d\rho(\vec{z}), \tag{4}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^l$.

Theorem 2.5 *Let $\Phi(x) = \phi((e_1,x),\cdots,(e_l,x))$ for w_φ-a.e. $x \in C[0,t]$, where ϕ is given by (4). Then for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[\Phi]$ exists and it is given by*

$$E^{anw_\lambda}[\Phi] = \int_{\mathbb{R}^l} \exp\left\{-\frac{1}{2\lambda}\|\vec{z}\|^2\right\} d\rho(\vec{z}).$$

Furthermore, for nonzero real q, $E^{anf_q}[\Phi]$ exists and it is given by the right hand side of the above equality where λ is replaced by $-iq$.

Proof. For $\lambda \in \mathbb{C}_+$, we have by (2), (4) and Theorem 2.4

$$\begin{aligned} E[\Phi^\lambda] &= \left(\frac{\lambda}{2\pi}\right)^{\frac{l}{2}} \int_{\mathbb{R}^l}\int_{\mathbb{R}^l} \exp\left\{i\langle\vec{u},\vec{z}\rangle - \frac{\lambda}{2}\|\vec{u}\|^2\right\} dm_L^l(\vec{u})d\rho(\vec{z}) \\ &= \int_{\mathbb{R}^l} \exp\left\{-\frac{1}{2\lambda}\|\vec{z}\|^2\right\} d\rho(\vec{z}). \end{aligned}$$

The theorem now follows from the dominated convergence theorem. □

Theorem 2.6 *Let $0 = t_0 < t_1 < \cdots < t_n \le t$ and G be given by*

$$G(x) = f(x(t_0), x(t_1), \cdots, x(t_n)), \text{ for } w_\varphi\text{-a.e. } x \in C[0,t]$$

where $f \in L_p(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}), m_L^n \otimes \varphi)\,(1 \le p \le \infty)$. Then for $\lambda \in \mathbb{C}_+$, $E^{anw_\lambda}[G]$ exists and it is given by

$$\begin{aligned} E^{anw_\lambda}[G] &= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}^n} f(u_0,u_1,\cdots,u_n) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n (u_j - u_{j-1})^2\right\} \\ &\quad dm_L^l(u_1,\cdots,u_n)d\varphi(u_0). \end{aligned}$$

Furthermore, for nonzero real q, $E^{anf_q}[G]$ exists if $p = 1$ and it is given by the right hand side the above equality where λ is replaced by $-iq$.

Proof. For $\lambda > 0$, we have by Lemma 1.1 and the change of variable theorem

$$\begin{aligned} E[G^\lambda] &= \int_C f(\lambda^{-\frac{1}{2}}(x(t_0), x(t_1), \cdots, x(t_n)))dw_\varphi(x) \\ &= \left(\frac{\lambda}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} f(u_0, u_1, \cdots, u_n) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{n}(u_j - u_{j-1})^2\right\} \\ & \quad dm_L^n(u_1, \cdots, u_n)d\varphi(u_0). \end{aligned}$$

The theorem now follows from the Morera's theorem and the dominated convergence theorem. □

3 Stability Theories

In this section, we introduce analogues of the analytic Feynman integrals of the functionals which are defined over $C^r[0, t]$, and of interested in Feynman integration theories itself and quantum mechanics. Most of their proofs are given in [4].

Let η be a complex valued Borel measure on $[0, t]$. Then $\eta = \mu + \nu$ can be decomposed uniquely into the sum of a continuous measure μ (with respect to the Lebesgue measure) and a discrete measure ν. Further, let δ_{p_j} denote the Dirac measure with total mass 1 concentrated at p_j.

Let $\mathcal{M}(\mathbb{R}^r)$ be the class of all complex Borel measures on $\mathbb{R}^r$ and $\mathcal{G}^*$ be the set of all $\mathbb{C}$-valued functions θ on $[0, \infty) \times \mathbb{R}^r$ which have the form

$$\theta(s, \vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{v}\rangle\} d\sigma_s(\vec{v}) \tag{5}$$

where $\{\sigma_s : s \in [0, \infty)\}$ is the family from $\mathcal{M}(\mathbb{R}^r)$ satisfying the following conditions;

(i) for each Borel subset E of $\mathbb{R}^r$, $\sigma_s(E)$ is a Borel measurable function of s on $[0, t]$,

(ii) $\|\sigma_s\| \in L_1([0, t], \mathcal{B}([0, t]), |\eta|)$.

Theorem 3.1 *Let m and k be two positive integers, and let $\eta = \mu + \sum_{j=1}^{m} w_j \delta_{p_j}$, where $0 < p_1 < \cdots < p_m < t$ and the w_js are in $\mathbb{C}$. Let $\theta \in \mathcal{G}^*$ be given by (5) and $F_k(x) = [\int_0^t \theta(s, x(s))d\eta(s)]^k$ for $x \in C^r[0, t]$. Then for $\lambda > 0$, we have*

$$E[F_k^\lambda] = \sum_{q_0+q_1+\cdots+q_m=k} \sum_{j_0+\cdots+j_m=q_0} H(k, \lambda, \vec{0}, q_0, q_1, \cdots, q_m; j_0, \cdots, j_m)$$

where for $\vec{v}_{m,j_m+1} \in \mathbb{R}^r$

$$\begin{aligned} & H(k,\lambda,\vec{v}_{m,j_m+1},q_0,q_1,\cdots,q_m;j_0,\cdots,j_m) \\ = \ & k!\Big(\prod_{\alpha=1}^{m}\frac{w_\alpha^{q_\alpha}}{q_\alpha!}\Big)\int_{\Delta_{q_0;j_0,\cdots,j_m}}\int_{\mathbb{R}^{kr}}\exp\Big\{-\frac{1}{2\lambda}\sum_{\alpha=0}^{m}\sum_{\beta=1}^{j_\alpha+1}(s_{\alpha,\beta}-s_{\alpha,\beta-1})\Big\|\sum_{\gamma=\beta}^{j_\alpha+1} \\ & \vec{v}_{\alpha,\gamma}+\sum_{l=\alpha+1}^{m}\sum_{\gamma=1}^{j_l+1}\vec{v}_{l,\gamma}\Big\|^2\Big\}\int_{\mathbb{R}^r}\exp\Big\{i\lambda^{-\frac{1}{2}}\Big\langle\vec{\eta},\sum_{\alpha=0}^{m}\sum_{\beta=1}^{j_\alpha+1}\vec{v}_{\alpha,\beta}\Big\rangle\Big\}d\varphi^r(\vec{\eta})d\Big(\prod_{\alpha=0}^{m} \\ & \prod_{\beta=1}^{j_\alpha}\sigma_{s_{\alpha,\beta}}\times\prod_{\alpha=1}^{m}\sigma_{p_\alpha}^{q_\alpha}\Big)(\vec{v},\vec{z})d\mu^{q_0}(\vec{s}) \end{aligned}$$

with $s_{0,0}=0$, $\vec{s}=(s_{0,1},\cdots,s_{0,j_0},s_{1,1},\cdots,s_{1,j_1},\cdots,s_{m,1},\cdots,s_{m,j_m})$, $s_{\alpha,0}=s_{\alpha-1,j_{\alpha-1}+1}=p_\alpha$ *for* $\alpha=1,\cdots,m$, $s_{m,j_m+1}=t$, $\Delta_{q_0;j_0,\cdots,j_m}=\{\vec{s}:0<s_{0,1}<\cdots<s_{0,j_0}<p_1<s_{1,1}<\cdots<s_{1,j_1}<p_2<\cdots<p_{m-1}<s_{m-1,1}<\cdots<s_{m-1,j_{m-1}}<p_m<s_{m,1}<\cdots<s_{m,j_m}<t\}$, $\vec{v}=(\vec{v}_{0,1},\cdots,\vec{v}_{0,j_0},\vec{v}_{1,1},\cdots,\vec{v}_{1,j_1},\cdots,\vec{v}_{m,1},\cdots,\vec{v}_{m,j_m})$ *and* $\vec{z}=(\vec{z}_{1,1},\cdots,\vec{z}_{1,q_1},\vec{z}_{2,1},\cdots,\vec{z}_{2,q_2},\cdots,\vec{z}_{m,1},\cdots,\vec{z}_{m,q_m})$; $\vec{v}_{\alpha-1,j_{\alpha-1}+1}=\sum_{l=1}^{q_\alpha}\vec{z}_{\alpha,l}$ *for* $\alpha=1,\cdots,m$.

Corollary 3.2 *Under the assumptions given as in Theorem 3.1 with one exception* $\eta=\mu$, *that is, assuming that* η *has no discrete part, we have*

$$\begin{aligned} E[F_k^\lambda] &= k!\int_{\Delta_k}\int_{\mathbb{R}^{kr}}\exp\Big\{-\frac{1}{2\lambda}\sum_{l=1}^{k}(s_l-s_{l-1})\Big\|\sum_{\gamma=l}^{k}\vec{v}_\gamma\Big\|^2\Big\}\int_{\mathbb{R}^r} \\ & \quad \exp\Big\{i\lambda^{-\frac{1}{2}}\Big\langle\vec{\eta},\sum_{l=1}^{k}\vec{v}_l\Big\rangle\Big\}d\varphi^r(\vec{\eta})d\Big(\prod_{l=1}^{k}\sigma_{s_l}\Big)(\vec{v})d\mu^k(\vec{s}) \end{aligned}$$

where $s_0=0$, $\vec{s}=(s_1,\cdots,s_k)$, $\vec{v}=(\vec{v}_1,\cdots,\vec{v}_k)$ *and* $\Delta_k=\{\vec{s}:0<s_1<\cdots<s_k<t\}$.

Corollary 3.3 *Under the assumptions given as in Theorem 3.1 with one exception* $\eta=\sum_{j=1}^{m}w_j\delta_{p_j}$, *that is, assuming that* η *has no continuous part, we have*

$$\begin{aligned} & E[F_k^\lambda] \\ = \ & k!\sum_{q_1+\cdots+q_m=k}\Big(\prod_{\alpha=1}^{m}\frac{w_\alpha^{q_\alpha}}{q_\alpha!}\Big)\int_{\mathbb{R}^{kr}}\exp\Big\{-\frac{1}{2\lambda}\sum_{\alpha=1}^{m}(p_\alpha-p_{\alpha-1})\Big\|\sum_{\gamma=\alpha}^{m}\sum_{l=1}^{q_\gamma}\vec{z}_{\gamma,l}\Big\|^2\Big\} \\ & \times\int_{\mathbb{R}^r}\exp\Big\{i\lambda^{-\frac{1}{2}}\Big\langle\vec{\eta},\sum_{\alpha=1}^{m}\sum_{l=1}^{q_\alpha}\vec{z}_{\alpha,l}\Big\rangle\Big\}d\varphi^r(\vec{\eta})d\Big(\prod_{\alpha=1}^{m}\sigma_{p_\alpha}^{q_\alpha}\Big)(\vec{z}) \end{aligned}$$

where $p_0=0$, $\vec{z}=(\vec{z}_{1,1},\cdots,\vec{z}_{1,q_1},\vec{z}_{2,1},\cdots,\vec{z}_{2,q_2},\cdots,\vec{z}_{m,1},\cdots,\vec{z}_{m,q_m})$.

Theorem 3.4 *Let φ^r be normally distributed with mean vector $\vec{0}$ and variance covariance matrix $\sigma^2 I_r$, where I_r is the r-dimensional identity matrix. Then, under the assumptions and notations given as in Theorem 3.1, $E^{anw_\lambda}[F_k]$ exists for $\lambda \in \mathbb{C}_+$ and it is given by*

$$E^{anw_\lambda}[F_k] \quad = \sum_{q_0+q_1+\cdots+q_m=k} \; \sum_{j_0+\cdots+j_m=q_0} T(k,\lambda,\sigma,\vec{0},q_0,\cdots,q_m;j_0,\cdots,j_m)$$

where for $\vec{v}_{m,j_m+1} \in \mathbb{R}^r$, $T(k,\lambda,\sigma,\vec{v}_{m,j_m+1},q_0,\cdots,q_m;j_0,\cdots,j_m)$ is given by the expression of $H(k,\lambda,\vec{v}_{m,j_m+1},q_0,q_1,\cdots,q_m;j_0,\cdots,j_m)$ replacing

$$\int_{\mathbb{R}^r} \exp\bigg\{ i\lambda^{-\frac{1}{2}} \bigg\langle \vec{\eta}, \sum_{\alpha=0}^{m} \sum_{\beta=1}^{j_\alpha+1} \vec{v}_{\alpha,\beta} \bigg\rangle \bigg\} d\varphi^r(\vec{\eta})$$

by

$$\exp\bigg\{ -\frac{\sigma^2}{2\lambda} \bigg\| \sum_{\alpha=0}^{m} \sum_{\beta=1}^{j_\alpha+1} \vec{v}_{\alpha,\beta} \bigg\|^2 \bigg\}.$$

Furthermore, for nonzero real q, $E^{anf_q}[F_k]$ exists and it is given by the above equality replacing λ by $-iq$.

Theorem 3.5 *Let the assumptions and notations be given as in Theorem 3.1 and let $F(x) = \exp\{\int_0^t \theta(s,x(s))d\eta(s)\}$ for $x \in C^r[0,t]$. Then for $\lambda > 0$, we have*

$$E[F^\lambda] = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} E[F_k^\lambda]$$

where $E[F_k^\lambda]$ is given as in Theorem 3.1. Furthermore, under the assumptions given as in Theorem 3.4, $E^{anf_q}[F]$ is obtained by

$$E^{anf_q}[F] = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} E^{anf_q}[F_k]$$

for nonzero real q, where $E^{anf_q}[F_k]$ is given as in Theorem 3.4.

Theorem 3.6 *Let the assumptions and notations be given as Theorem 3.1 and let $G_k(x) = F_k(x)\psi(x(t))$ for $x \in C^r[0,t]$, where*

$$\psi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle \vec{u}, \vec{v}\rangle\} d\nu(\vec{v})$$

for $\nu \in \mathcal{M}(\mathbb{R}^r)$. *Let* $G(x) = \exp\{\int_0^t \theta(s, x(s))d\eta(s)\}\psi(x(t))$. *Then for* $\lambda > 0$, *we have*

$$E[G^\lambda] = \int_{\mathbb{R}^r}\int_{\mathbb{R}^r} \exp\left\{i\lambda^{-\frac{1}{2}}\langle\vec{\eta}, \vec{v}\rangle - \frac{t}{2\lambda}\|\vec{v}\|^2\right\}d\varphi^r(\vec{\eta})d\nu(\vec{v}) + \sum_{k=1}^{\infty}\frac{1}{k!}E[G_k^\lambda]$$

where

$$\begin{aligned} E[G_k^\lambda] \;=\; & \sum_{q_0+q_1+\cdots+q_m=k}\ \sum_{j_0+\cdots+j_m=q_0}\int_{\mathbb{R}^r} H(k, \lambda, \\ & \vec{v}_{m,j_m+1}, q_0, q_1, \cdots, q_m; j_0, \cdots, j_m)d\nu(\vec{v}_{m,j_m+1}). \end{aligned}$$

Furthermore, under the assumptions and notations given as in Theorem 3.4, $E^{anf_q}[G]$ *can be obtained by*

$$E^{anf_q}[G] = \int_{\mathbb{R}^r}\exp\left\{\frac{t+\sigma^2}{2qi}\|\vec{v}\|^2\right\}d\nu(\vec{v}) + \sum_{k=1}^{\infty}\frac{1}{k!}E^{anf_q}[G_k]$$

for nonzero real q, *where*

$$\begin{aligned} E^{anf_q}[G_k] \;=\; & \sum_{q_0+q_1+\cdots+q_m=k}\ \sum_{j_0+\cdots+j_m=q_0}\int_{\mathbb{R}^r} T(k, -iq, \sigma, \vec{v}_{m,j_m+1}, \\ & q_0, \cdots, q_m; j_0, \cdots, j_m)d\nu(\vec{v}_{m,j_m+1}). \end{aligned}$$

Corollary 3.7 *Under the assumptions and notations given as in Theorem 3.6 with one exception* $\varphi^r = \delta_{\vec{0}}$, *the Dirac measure concentrated at* $\vec{0} \in \mathbb{R}^r$, *we have for a nonzero real* q

$$E^{anf_q}[G] = \int_{\mathbb{R}^r}\exp\left\{\frac{t}{2qi}\|\vec{v}\|^2\right\}d\nu(\vec{v}) + \sum_{k=1}^{\infty}\frac{1}{k!}E^{anf_q}[G_k]$$

where

$$\begin{aligned} E^{anf_q}[G_k] \;=\; & \sum_{q_0+q_1+\cdots+q_m=k}\ \sum_{j_0+\cdots+j_m=q_0}\int_{\mathbb{R}^r} T(k, -iq, 0, \\ & \vec{v}_{m,j_m+1}, q_0, \cdots, q_m; j_0, \cdots, j_m)d\nu(\vec{v}_{m,j_m+1}) \end{aligned}$$

which is a main result of [8].

Remark 3.8

- *Under the conditions given as in Corollaries 3.2 and 3.3, we can obtain more simple expressions in Theorems 3.4, 3.5, 3.6 and Corollary 3.7.*
- *If* $\eta = \mu + \sum_{j=1}^m w_j\delta_{p_j}$, *where* $0 \le p_1 < \cdots < p_m \le t$, *we can obtain all the results in the present section with minor modifications.*

- *If* $\eta = \mu + \sum_{j=1}^{\infty} w_j \delta_{p_j}$*, then using the following version of the* $\aleph_0$*-nomial formula [7, p.41]*

$$\left(\sum_{p=0}^{\infty} b_p\right)^n = \sum_{h=0}^{\infty} \sum_{q_0+q_1+\cdots+q_h=n, q_h\neq 0} \frac{n!}{q_0!q_1!\cdots q_h!} b_0^{q_0} b_1^{q_1} \cdots b_h^{q_h},$$

we can show that for $\lambda > 0$*,* $E[G^\lambda]$ *exists in Theorem 3.6.*

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