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Spacelike Salkowski and anti-Salkowski curves with timelike principal normal in Minkowski 3-space

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Abstract

A century ago, Salkowski [1] introduced a family of curves with constant curvature but non-constant torsion (Salkowski curves) and a family of curves with constant torsion but non-constant curvature (anti-Salkowski curves). Ali (2009–2010) [2], [3] adapted the definition of such curves in Minkowski 3-space and introduced an explicit parametrization of a timelike and a spacelike (with a spacelike principal normal vector) Salkowski and anti-Salkowski curves. In this paper, we introduce an explicit parametrization of a spaelike Salkowski and anti-Salkowski curves with a timelike principal normal vector in Minkowski 3-space. Moreover, we characterize them as a space curve with constant curvature or constant torsion and whose normal vector makes a constant angle with a fixed straight line.

Mathematics Subject Classification: 53C40, 53C50

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1 Introduction

Salkowski (resp. anti-Salkowski) curves in Euclidean space $\mathbf{E}^3$ are generally known as family of curves with constant curvature (resp. torsion) but non-constant torsion (resp. curvature) with an explicit parametrization. They were defined in an earlier paper [1] and retrieved, as an example of tangentially cubic curves [4], in a first version of Pottmann and Hofer [5]. Recently, Monterde [6] studied some of characterizations of these curves and he prove that the normal vector makes a constant angle with a fixed straight line. In (2009–2010), Ali [2], [3] adapted the definition of such curves in Minkowski 3-space. Also, he introduced an explicit parametrization of a timelike and a spacelike (with a spacelike principal normal vector) Salkowski and anti-Salkowski curves.

Analogously, in this paper, we introduce the explicit parametrization of a spacelike Salkowski and anti-Salkowski curves with a timelike principal normal vector in Minkowski space $\mathbf{E}_1^3$ and we study some characterizations of these curves.

2 Preliminaries

First, we briefly present theory of the curves in Minkowski 3-space as follows:

The Minkowski three-dimensional space $\mathbf{E}_1^3$ is the real vector space $\mathbf{R}^3$ endowed with the standard flat Lorentzian metric given by:

$$\langle , \rangle = dx_1^2 + dx_2^2 - dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}_1^3$. If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are arbitrary vectors in $\mathbf{E}_1^3$, we define the (Lorentzian) vector product of $\mathbf{u}$ and $\mathbf{v}$ as the following:

$$u \times v = -\begin{vmatrix} i & j & -k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

An arbitrary vector $\mathbf{v} \in \mathbf{E}_1^3$ is said to be a spacelike if $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ or $\mathbf{v} = 0$, timelike if $\langle \mathbf{v}, \mathbf{v} \rangle < 0$, and lightlike (or null) if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ and $\mathbf{v} \neq 0$. The norm (length) of a vector $\mathbf{v}$ is given by $\| \mathbf{v} \| = \sqrt{|\langle \mathbf{v}, \mathbf{v} \rangle|}$. An arbitrary regular (smooth) curve $\alpha : I \subset \mathbf{R} \to \mathbf{E}_1^3$ is locally spacelike if all of its velocity vectors $\alpha'(t)$ are spacelike for each $t \in I \subset \mathbf{R}$. If α is spacelike, there exists a change of the parameter t, namely, $s = s(t)$, such that $\| \alpha'(s) \| = 1$. We say then that α is a unit speed curve [7], [8], [9], [10], [11], [12], [13].

Given a unit speed curve α in Minkowski space $\mathbf{E}_1^3$ it is possible to define a Frenet frame $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ associated for each point s [14], [15]. Here $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ are the tangent, principal normal and binormal vector field, respectively.

Now and in the next, we suppose that α is a spacelike curve with a timelike principal normal vector $\mathbf{N}$. Then $\mathbf{T}'(s) \neq 0$ is a spacelike vector independent with $\mathbf{T}(s)$. We define the curvature of α at s as $\kappa(s) = |\mathbf{T}'(s)|$. The principal normal vector $\mathbf{N}(s)$ and the binormal vector $\mathbf{B}(s)$ are defined as [16]:

$$\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\kappa(s)} = \frac{\alpha''}{|\alpha''|}, \ \ \mathbf{B}(s) = -\mathbf{T}(s) \times \mathbf{N}(s),$$

where the vector $\mathbf{N}(s)$ is unitary and timelike. For each s, $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an orthonormal base of $\mathbf{E}_1^3$ which is called the Frenet trihedron of α. We define the torsion of α at s as:

$$\tau(s) = \langle \mathbf{N}'(s), \mathbf{B}(s) \rangle.$$

Then the Frenet formula is

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}, \tag{1}$$

where

$$\langle \mathbf{T}, \mathbf{T} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1, \langle \mathbf{N}, \mathbf{N} \rangle = -1, \langle \mathbf{T}, \mathbf{N} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = \langle \mathbf{B}, \mathbf{T} \rangle = 0.$$

3 Spacelike Salkowski curves with a timelike principal normal

In this section, we introduce the explicit parametrization of a spacelike Salkowski curves with a timelike principal normal vector in Minkowski space $\mathbf{E}_1^3$ as the following:

Definition 3.1 *For any $m \in R$ with $m > 1$ or $m < -1$, let us define the space curve*

$$\gamma_m(t) = \frac{n}{4m}\Big(2\sin[t] - \tfrac{1+n}{1-2n}\sin[(1-2n)t] - \tfrac{1-n}{1+2n}\sin[(1+2n)t], \; 2\cos[t] - \tfrac{1+n}{1-2n}\cos[(1-2n)t] - \tfrac{1-n}{1+2n}\cos[(1+2n)t], \; \tfrac{1}{m}\cos[2nt] \Big), \tag{2}$$

with $n = \frac{m}{\sqrt{m^2-1}}$.

We will call a spacelike Salkowski curve with a timelike principal normal vector in Minkowski space $\mathbf{E}_1^3$. One can see a special examples of such curves in the (positive case of m) figure 1 and in the (negative case of m) figure 2.

The geometric elements of this curve γ_m are the following:
(1): $\langle \gamma_m', \gamma_m' \rangle = \frac{\sin^2[nt]}{m^2-1}$, so $\|\gamma_m'\| = \frac{\sin[nt]}{\sqrt{m^2-1}}$
(2): The arc-length parameter is $s = -\frac{\cos[nt]}{m}$.
(3): The curvature $\kappa(t) = 1$ and the torsion $\tau(t) = \cot[nt]$.
(4): The Frenet frame is

$$\begin{aligned} \mathbf{T}(t) = \ & \Big(\cos[t]\sin[nt] - n\sin[t]\cos[nt], \\ & -\sin[t]\sin[nt] - n\cos[t]\cos[nt], -\tfrac{n}{m}\cos[nt]\Big), \\ \mathbf{N}(t) = \ & \tfrac{n}{m}\Big(\sin[t], \cos[t], m\Big), \\ \mathbf{B}(t) = \ & \Big(-\cos[t]\cos[nt] - n\sin[t]\sin[nt], \\ & \sin[t]\cos[nt] - n\cos[t]\sin[nt], -\tfrac{n}{m}\sin[nt]\Big). \end{aligned} \tag{3}$$

From the expression of the normal vector, see Equation (3), we can see that the normal indicatrix, or nortrix, of a Salkowski curve (2) in Minkowski space $\mathbf{E}_1^3$ describes a parallel of the unit sphere. The hyperbolic angle between the timelike normal vector $\mathbf{N}$ and the timelike vector $(0, 0, -1)$ is constant and equal to $\phi = \pm\text{arccosh}[n]$. This fact is reminiscent of what happens with another important class of curves, the general helices in Minkowski space $\mathbf{E}_1^3$. Such a condition implies that the tangent indicatrix, or tantrix, describes a parallel in the unit sphere.

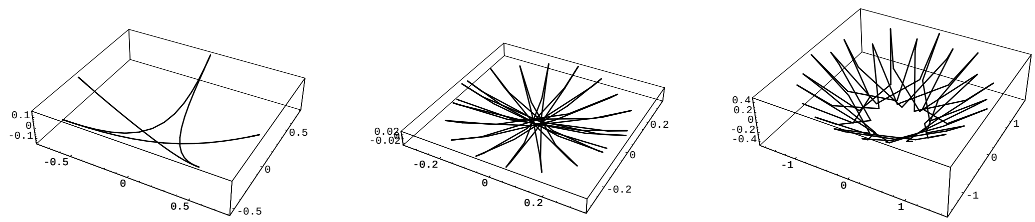

Figure 1: Some Salkowski curves for $m = \frac{3}{2}, 3, \frac{10}{9}$.

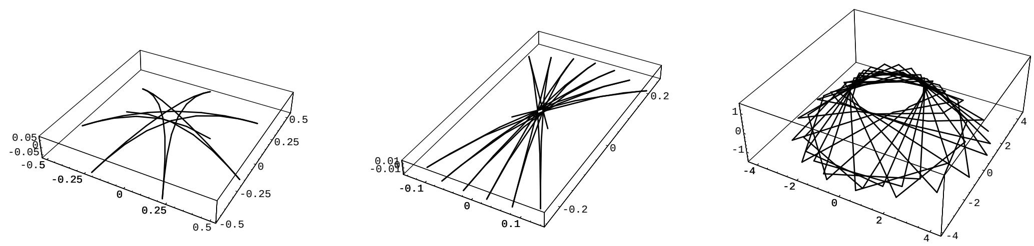

Figure 2: Some Salkowski curves for $m = -2, -4, -\frac{70}{69}$.

Lemma 3.2 *Let $\alpha : I \to \mathbf{E}_1^3$ be a spacelike curve with a timelike principal normal vector parameterized by arc-length with $\kappa = 1$. The normal vector make a constant hyperbolic angle, ϕ, with a fixed straight line in space if and only if $\tau(s) = \pm\frac{s}{\sqrt{\tanh^2[\phi]-s^2}}$.*

proof: ($\Rightarrow$) Let $\mathbf{d}$ be the unitary timelike fixed vector makes a constant hyperbolic angle ϕ with the timelike normal vector $\mathbf{N}$. Therefore

$$\langle \mathbf{N}, \mathbf{d} \rangle = \cosh[\phi]. \tag{4}$$

Differentiating Equation (4) and using Frenet's equations (1), we get

$$\langle \mathbf{T} + \tau \mathbf{B}, \mathbf{d} \rangle = 0. \tag{5}$$

Therefore,

$$\langle \mathbf{T}, \mathbf{d} \rangle = -\tau \langle \mathbf{B}, \mathbf{d} \rangle.$$

If we put $\langle \mathbf{B}, \mathbf{d} \rangle = -b$, we can write

$$\mathbf{d} = \tau\, b\, \mathbf{T} + \cosh[\phi]\mathbf{N} - b\, \mathbf{B}.$$

From the unitary of the vector $\mathbf{d}$ we get $b = \pm\frac{\sinh[\phi]}{\sqrt{1+\tau^2}}$. Therefore, the vector $\mathbf{d}$ can be written as

$$\mathbf{d} = \pm\frac{\tau\, \sinh[\phi]}{\sqrt{1+\tau^2}}\mathbf{T} + \cosh[\phi]\mathbf{N} \mp \frac{\sinh[\phi]}{\sqrt{1+\tau^2}}\mathbf{B}. \tag{6}$$

If we differentiate Equation (5) again, we obtain

$$\langle \dot{\tau}\mathbf{B} + (1+\tau^2)\mathbf{N}, \mathbf{d} \rangle = 0. \tag{7}$$

Equations (6) and (7) lead to the following differential equation

$$\pm \tanh[\phi]\frac{\dot{\tau}}{(1+\tau^2)^{3/2}} + 1 = 0.$$

Integration the above equation, we get

$$\pm \tanh[\phi]\frac{\tau}{\sqrt{1+\tau^2}} + s + c = 0. \tag{8}$$

where c is an integration constant. The integration constant can disappear with a parameter change $s \to s - c$. Finally, to solve (8) with τ as unknown we express the desired result.

($\Leftarrow$) Suppose that $\tau = \pm\frac{s}{\sqrt{\tanh^2[\phi]-s^2}}$ and let us consider the timelike vector

$$\mathbf{d} = \cosh[\phi]\Big(- s\, \mathbf{T} + \mathbf{N} \mp \sqrt{\tanh^2[\phi] - s^2}\, \mathbf{B}\Big).$$

We will prove that the vector $\mathbf{d}$ is a constant vector. Indeed, applying Frenet formula

$$\dot{\mathbf{d}} = \cosh[\phi]\Big(-\mathbf{T} - s\mathbf{N} + \mathbf{T} + \tau\mathbf{B} \mp \frac{s}{\sqrt{\tanh^2[\phi] - s^2}}\mathbf{B} \pm \tau\sqrt{\tanh^2[\phi] - s^2}\mathbf{N}\Big) = 0$$

Therefore, the vector $\mathbf{d}$ is constant and $\langle \mathbf{N}, \mathbf{d}\rangle = \cosh[\phi]$. This concludes the proof of Lemma (3.2).

Once the intrinsic or natural equations of a curve have been determined, the next step is to integrate Frenet formula with $\kappa = 1$ and

$$\tau = \pm\frac{s}{\sqrt{\tanh^2[\phi] - s^2}} = \mp\frac{-\frac{s}{\tanh[\phi]}}{\sqrt{1 - \left(\frac{s}{\tanh[\phi]}\right)^2}}.$$

If we put $\cos[\theta] = -\frac{s}{\tanh[\phi]}$, the equation takes the form

$$\tau = \mp\cot[\theta] = \mp\cot\Big[\arccos[-\frac{s}{\tanh[\phi]}]\Big]. \tag{9}$$

Theorem 3.3 *A spacelike curve has a timelike principal normal vector in Minkowski space $\mathbf{E}_1^3$ with $\kappa = 1$ and such that their normal vector makes a constant angle with a fixed straight line is, up a rigid motion of the space or up to the antipodal map, $p \to -p$, spacelike Salkowski curve with a timelike principal normal vector.*

Proof: We know from Definition 3.1 that the arc-length parameter of a Salkowski curve (2) is $s = \int_0^t \|\gamma_m'(u)\| du = -\frac{1}{m}\cos[nt]$. Therefore, $t = \frac{1}{n}\arccos[-ms]$. In terms of the arc-length curvature and torsion are then

$$\kappa(s) = 1, \quad \tau(s) = \cot[\arccos[-ms]],$$

the same intrinsic equations, with $m = \coth[\phi]$ and $n = \frac{m}{\sqrt{m^2-1}} = \cosh[\phi]$ (compare with the positive case in Equation (9)), as the ones shown in Lemma 3.2.

For the negative case in Equation (9), let us recall that if a curve α has torsion τ_α, then the curve $\beta(t) = -\alpha(t)$ has as torsion $\tau_\beta(t) = -\tau_\alpha(t)$, whereas curvature is preserved.

Therefore, the fundamental theorem of curves in Minkowski space states in our situation that, up a rigid motion or up to the antipodal map, the curves we are looking for are spacelike Minkowski curves with a timelike principal normal vector.

4 Spacelike anti-Salkowski curves with a timelike principal normal

As an additional material we will show in this section how to build, from a curve in Minkowski space $\mathbf{E}_1^3$ of constant curvature, another curve of constant torsion.

Let us recall that a curve $\alpha : I \to \mathbf{E}_1^3$, is 2-regular at a point t_0 if $\alpha'(t_0) \neq 0$ and if $\kappa_\alpha(t_0) \neq 0$.

Lemma 4.1 *Let $\alpha : I \to \mathbf{E}_1^3$ be a regular spacelike curve with a timelike principal normal vector parameterized by arc-length with curvature κ_α, torsion τ_α and Frenet frame $\{\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha\}$. Let us $\beta(t) = \int_0^t \mathbf{T}_\alpha(u)\|\mathbf{B}_\alpha'(u)\|\, du$. If $s_\alpha \in I$ satisfies $\tau_\alpha(s_\alpha) \neq 0$, the curve β is 2-regular at s_β and*

$$\kappa_\beta = \frac{\kappa_\alpha}{\tau_\alpha}, \;\; \tau_\beta = 1, \;\; \mathbf{T}_\beta = \mathbf{T}_\alpha, \;\; \mathbf{N}_\beta = \mathbf{N}_\alpha, \;\; \mathbf{B}_\beta = \mathbf{B}_\alpha.$$

Proof: In order to obtain the tangent vector of β let us compute

$$\mathbf{T}_\beta(s_\beta) = \dot{\beta}(s_\beta) = \frac{d\beta}{dt}\frac{dt}{ds_\beta} = \mathbf{T}_\alpha\|\mathbf{B}_\alpha'(t)\|\frac{dt}{ds_\beta}.$$

From the above equation, we get

$$\frac{ds_\beta}{dt} = \|\mathbf{B}_\alpha'(t)\| = \Big\|\frac{\mathbf{B}_\alpha}{ds_\alpha}\frac{ds_\alpha}{dt}\Big\| = \tau_\alpha\frac{ds_\alpha}{dt}, \tag{10}$$

and

$$\mathbf{T}_\beta(s_\beta) = \mathbf{T}_\alpha(s_\alpha).$$

Differentiation the above equation using Frenet's Equations (1) we obtain

$$\dot{\mathbf{T}}_\beta(s_\beta) = \frac{d\mathbf{T}_\alpha}{ds_\alpha}\,\frac{ds_\alpha}{dt}\,\frac{dt}{ds_\beta}.$$

Using Frenet's Equations (1) and Equation (10), the above equation writes

$$\kappa_\beta\,\mathbf{N}_\beta(s_\beta) = \frac{\kappa_\alpha}{\tau_\alpha}\mathbf{N}_\alpha(s_\alpha)$$

From the above equation, we get

$$\kappa_\beta = \frac{\kappa_\alpha}{\tau_\alpha},$$

and

$$\mathbf{N}_\beta(s_\beta) = \mathbf{N}_\alpha(s_\alpha).$$

So we have

$$\mathbf{B}_\beta(s_\beta) = -\mathbf{T}_\beta(s_\beta) \times \mathbf{N}_\beta(s_\beta) = -\mathbf{T}_\alpha(s_\alpha) \times \mathbf{N}_\alpha(s_\alpha) = \mathbf{B}_\alpha(s_\alpha).$$

Differentiating the above equation with respect to s_β we get $\tau_\beta = 1$.

Let us apply the previous result to the curve γ_m defined in Equation (2) we have the explicit parametrization of an anti-Salkowski curve as follows:

$$\beta_m(t) = \frac{n}{4m}\Bigg(\begin{array}{l} 2n\cos[t] - \frac{1-n}{1+2n}\cos[(1+2n)t] + \frac{1+n}{1-2n}\cos[(1-2n)t], \\ +2n\sin[t] - \frac{1-n}{1+2n}\sin[(1+2n)t] + \frac{1+n}{1-2n}\sin[(1-2n)t], \\ -\frac{1}{m}(2nt + \sin[2nt])\Bigg), \end{array} \tag{11}$$

where $n = \frac{m}{\sqrt{m^2-1}}$. Let us call these curves by the name spacelike anti-Salkowski curves with a timelike principal normal vector. The presence of the non-trigonometric term $2nt$ in the third component of β_m makes that the change of variable studied in Section 2 for Salkowski curves does not work for anti-Salkowski. Moreover, an examples of such curves can be seen in the figure 3.

Applying Lemma 4.1 we get the following

Proposition 4.2 *The curves β_m in Equation (11) are curves of constant torsion equal to 1 and non-constant curvature equal to* $\tan[nt]$.

Finally, we state here the following:

Lemma 4.3 *Let $\alpha : I \to \mathbf{E}_1^3$ be a regular spacelike curve with a timelike principal normal vector parameterized by arc-length with curvature κ_α, torsion τ_α and Frenet frame $\{\mathbf{T}_\alpha, \mathbf{N}_\alpha, \mathbf{B}_\alpha\}$. Let us consider the curve $\beta(t) = \int_0^t \mathbf{T}_\alpha(u)\|\mathbf{T}_\alpha'(u)\|\,du$. Then at a parameter $s_\alpha \in I$ such that $\kappa_\alpha(s_\alpha) \neq 0$, the curve β is 2-regular at s_β and*

$$\kappa_\beta = 1,\ \ \tau_\beta = \frac{\tau_\alpha}{\kappa_\alpha},\ \ \mathbf{T}_\beta = \mathbf{T}_\alpha,\ \ \mathbf{N}_\beta = \mathbf{N}_\alpha,\ \ \mathbf{B}_\beta = \mathbf{B}_\alpha.$$

Proof: The proof of this Lemma is similar as the proof of Lemma 4.1.

Theorem 4.4 *The spacelike curve with a timelike principal normal vector and $\tau = 1$ such that their principal normal vectors make a constant hyperbolic angle with a fixed straight line are the spacelike anti-Salkowski curves defined in Equation (11).*

Proof: Let α be a spacelike curve has a timelike principal normal vector with $\tau = 1$ and let $\beta(t) = \int_0^t \mathbf{T}_\alpha(u)\|\mathbf{T}'_\alpha(u)\|\,du$. By Lemma 4.3, β is a curve with constant curvature $\kappa = 1$, non-constant torsion $\tau = \frac{1}{\kappa_\alpha}$ and with the same principal normal vector. Therefore, β is a Salkowski curve and α is an anti-Salkowski curve in Minkowski 3-space.

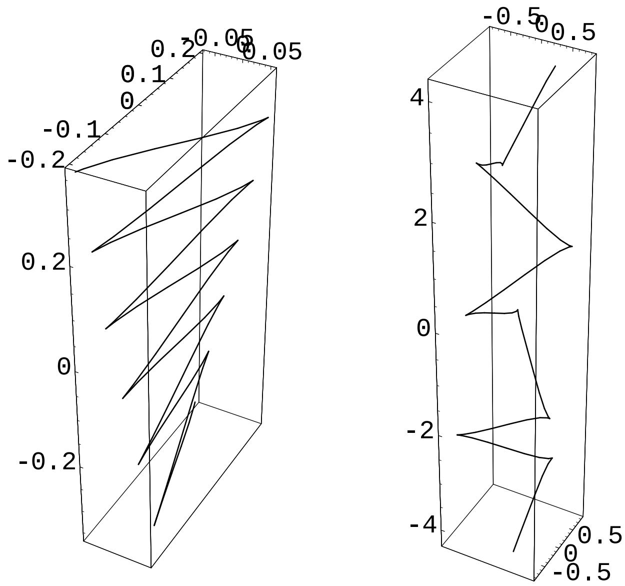

Figure 3: Some anti-Salkowski curves for $m = 5$ and $m = -\frac{3}{2}$.

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