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Improved bounds for some nonstandard problems for Maxwell-Cattaneo equations

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Abstract

We consider the Maxwell-Cattaneo equations where the temperature θ and heat flux $\mathbf{u}$ satisfy a non-standard auxiliary condition which prescribes a combination of their values initially. The L_2 bound for temperature is obtained by using Lagrange identities.

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1 Introduction

In this paper, we consider the following equations

$$\tau u_{i,t} = u_i - k\theta_{,i} + \mu\Delta u_i + \gamma u_{j,ji} \ ; \tag{1.1}$$

$$k\theta_{,ii} = u_{i,i} + c\dot{\theta}, \tag{1.2}$$

with boundary conditions

$$u_i = 0, \quad \nabla\theta = 0, \quad \theta = 0, \quad on\ \partial\Omega \times [0, T], \tag{1.3}$$

and the nonstandard conditions

$$\theta(x,T)+\alpha\theta(x,0)=g(x), \quad u_i(x,T)+\alpha u_i(x,0)=f_i(x),\ x\in\Omega \tag{1.4}$$

for some constant α.

We take the divergence of (1.1) and eliminate the heat flux terms to arrive at the following initial-boundary value problem for the temperature θ:

$$\tau\theta_{,tt}+\theta_{,t}-a\Delta\theta_{,t}-2kc^{-1}\Delta\theta+b\Delta^2\theta=0 \quad in\ \Omega\times(0,T) \tag{1.5}$$

$$\nabla\theta=0, \quad \theta_{,t}=0, \quad \theta=0 \quad on\ \partial\Omega\times[0,T]; \tag{1.6}$$

$$\theta(x,T)+\alpha\theta(x,0)=g(x) \quad in\ \Omega; \tag{1.7}$$

$$\theta_{,t}(x,T)+\alpha\theta_{,t}(x,0)=h(x) \quad in\ \Omega, \tag{1.8}$$

where $a=c^{-1}[\tau k+(\mu+\gamma)c], b=c^{-1}(\mu+\gamma)k$ and $h=c^{-1}(k\Delta g-f_{i,i})$. For compatibility, we assume that g vanishes on $\partial\Omega$. In fact, there are many papers to study the nonstandard problems for many equations. More references, one could refer to [1]-[8].

In the present paper,the comma is used to indicate partial differentiation and the differentiation with respect to the direction x_k is denoted as $,k$,thus $u_{,i}$ denotes $\frac{\partial u}{\partial x_i}$.The usual summation convection is employed with repeated Latin subscripts i summed from 1 to 3.Hence, $u_{i,i}=\sum\limits_{i=1}^{3}\frac{\partial u_i}{\partial x_i}$.

2 Bounds for the temperature θ

In this section we consider the initial-boundary problem (1.5)-(1.8) and seek L_2 bounds for θ which is valid in the interval (0,1).

To this end, we set

$$\widetilde{\theta}(x,t)=\theta(x,2t-\eta)$$

and use Lagrange identity for $0\le t\le\frac{T}{2}$,

$$\begin{aligned}\int_0^t\int_\Omega&\widetilde{\theta}_\eta(\tau\theta_{,\eta\eta}+\theta_{,\eta}-a\Delta\theta_{,\eta}-2kc^{-1}\Delta\theta+b\Delta^2\theta)\\&+\theta_{,\eta}(\tau\theta_{,\eta\eta}-\theta_{,\eta}+a\Delta\theta_{,\eta}-2kc^{-1}\Delta\theta+b\Delta^2\theta)dxd\eta=0\end{aligned} \tag{2.1}$$

from which it follows that

$$\int_0^t\int_\Omega\tau(\widetilde{\theta}_{,\eta}\theta_{,\eta})_{,\eta}+2kc^{-1}(\theta_{,i\eta}\widetilde{\theta}_{,i}+\theta_{,i}\widetilde{\theta}_{,i\eta})+b(\Delta\widetilde{\theta}_{,\eta}\Delta\theta+\Delta\theta_{,\eta}\Delta\widetilde{\theta})dxd\eta=0, \tag{2.2}$$

where we have used the divergence theorem.
We define

$$P(t)=\tau\int_\Omega \theta_{,t}(t)\theta_{,t}(t)dx+2kc^{-1}\int_\Omega \theta_{,i}(t)\theta_{,i}(t)dx+b\int_\Omega \theta(t)\Delta\theta(t)dx \tag{2.3}$$

and from (2.2) we know

$$P(t)=\tau\int_\Omega \theta_{,t}(2t)\theta_{,t}(0)dx+2kc^{-1}\int_\Omega \theta_{,i}(0)\theta_{,i}(2t)dx+b\int_\Omega \Delta\theta(2t)\Delta\theta(0)dx. \tag{2.4}$$

We now compute

$$\begin{aligned}\frac{dP(t)}{dt}&=2\int_\Omega \theta_{,t}(-\theta_{,t}+a\Delta\theta_{,t}+2kc^{-1}\Delta\theta-b\Delta^2\theta)dx+4kc^{-1}\int_\Omega \theta_{,i}\theta_{,it}dx+2b\int_\Omega \Delta\theta\Delta\theta_{,t}dx\\&\le -2\int_\Omega \theta_{,t}\theta_{,t}dx-2a\int_\Omega \theta_{,it}\theta_{,it}dx.\end{aligned} \tag{2.5}$$

Obviously, we deduce that $P(t)$ is non-increasing on the interval [0,T]. Hence, we obtain

$$P(T)\le P(\frac{T}{2}),\quad P(t)\le P(0),\quad 0\le t\le T. \tag{2.6}$$

In order to seek the bound for $P(t)$, we need bound $P(0)$ firstly. To this end we use $(2.6)^1$ along with (1.8), and (1.7) to write

$$\begin{aligned}P(T)&=\tau\int_\Omega \theta_{,t}(T)\theta_{,t}(T)dx+2kc^{-1}\int_\Omega \theta_{,i}(T)\theta_{,i}(T)dx+b\int_\Omega \Delta\theta(T)\Delta\theta(T)dx\\&=\tau\int_\Omega (h-\alpha\theta_{,t}(0))(h-\alpha\theta_{,t}(0))dx+2kc^{-1}\int_\Omega (g_{,i}-\alpha\theta_{,i}(0))(g_{,i}-\alpha\theta_{,i}(0))dx\\&+b\int_\Omega (\Delta g-\alpha\Delta\theta(0))(\Delta g-\alpha\Delta\theta(0))dx\\&=\tau\int_\Omega h^2dx+\alpha^2\int_\Omega \theta_{,t}^2(0)dx-2\alpha\tau\int_\Omega \theta_{,t}(0)hdx\\&+2kc^{-1}\int_\Omega g_{,i}g_{,i}dx+2kc^{-1}\alpha^2\int_\Omega \theta_{,i}(0)\theta_{,i}(0)dx-4kc^{-1}\alpha\int_\Omega g_{,i}\theta_{,i}(0)dx\\&+b\int_\Omega \Delta g\Delta gdx+\alpha^2\int_\Omega \Delta\theta(0)\Delta\theta(0)dx-2b\alpha\int_\Omega \Delta\theta(0)\Delta gdx\end{aligned} \tag{2.7}$$

and

$$
\begin{aligned}
P(\frac{T}{2}) &= \int_\Omega \theta_{,t}(T)\theta_{,t}(0)dx + 2kc^{-1}\int_\Omega \theta_{,i}(T)\theta_{,i}(0)dx + b\int_\Omega \Delta\theta(0)\Delta\theta(T)dx \\
&= \tau\int_\Omega \theta_{,t}(0)hdx - \alpha\tau\int_\Omega \theta_{,t}^2(0)dx + 2kc^{-1}\int_\Omega g_{,i}\theta_{,i}(0)dx - 2kc^{-1}\alpha\int_\Omega \theta_{,i}(0)\theta_{,i}(0)dx \\
&+ b\int_\Omega \Delta\theta(0)\Delta g dx - b\alpha\int_\Omega \Delta\theta(0)\Delta\theta(0)dx.
\end{aligned}
\tag{2.8}
$$

We set

$$H = \tau\int_\Omega h^2 dx + 2kc^{-1}\int_\Omega g_{,i}g_{,i}dx + b\int_\Omega \Delta g\Delta g dx,$$

and use the inequality

$$m_1n_1 + m_2n_2 \le (m_1+m_2)^{\frac{1}{2}}(n_1+n_2)^{\frac{1}{2}}$$

for positive constants m_1, m_2, n_1, n_2. Then we have

$$
\begin{aligned}
(\alpha^2+\alpha)P(0) + H &\le\mid 2\alpha+1 \mid [\tau\int_\Omega \theta_{,t}(0)hdx + 2kc^{-1}\int_\Omega g_{,i}\theta_{,i}(0)dx + b\int_\Omega \Delta\theta(0)\Delta g dx] \\
&\le\mid 2\alpha+1 \mid [P(0)]^{\frac{1}{2}}H^{\frac{1}{2}},
\end{aligned}
\tag{2.9}
$$

it follows that

$$P^{\frac{1}{2}}(0) \le \frac{\mid 2\alpha+1\mid +1}{2(\alpha^2+\alpha)}H^{\frac{1}{2}}, \quad 0\le t\le T \tag{2.10}$$

provided $\alpha^2+\alpha>0$ i.e.,for $\alpha>0$ or $\alpha<-1$. Consequently, we have

$$P(0) \le \frac{(\mid 2\alpha+1\mid +1)^2}{4(\alpha^2+\alpha)^2}H, \quad 0\le t\le T.$$

In light of $(2.9)^2$, we obtain

$$P(t) \le \frac{(\mid 2\alpha+1\mid +1)^2}{4(\alpha^2+\alpha)^2}H, \quad 0\le t\le T. \tag{2.11}$$

From (2.14) and (2.3) we can get

$$\int_\Omega \theta^2 dx \le \frac{1}{2kc^{-1}\lambda}\frac{(\mid 2\alpha+1\mid +1)^2}{4(\alpha^2+\alpha)^2}H \doteqdot B_0, \quad 0\le t\le T. \tag{2.12}$$

Since $\theta(x,t), \theta_i(x,t)$ vanish on $\partial\Omega$, it follows that

$$\int_\Omega \theta_{,i}\theta_{,i}dx \ge \lambda\int_\Omega \theta^2 dx \tag{2.13}$$

where λ is the first eigenvalue of the membrane problem

$$\begin{aligned} \Delta\varphi + \varphi &= 0 \quad in\ \Omega; \\ \varphi &= 0 \quad on\ \partial\Omega. \end{aligned} \tag{2.14}$$

So we have established the following theorem:

Theorem 1: Let $\theta(x,t)$ be a classical solution of (1.5)-(1.8), then provided α satisfied $\alpha^2 + \alpha > 0$, $P(t)$ is bounded by (2.11) and the L_2 integral of θ by (2.12),where $P(t)$ is defined by (2.3).

We note if $\alpha > 0$, we have simpler bound

$$P(t) \leq \frac{H}{\alpha^2}, \quad \int_\Omega \theta^2 dx \leq \frac{b\lambda^{-2}}{\alpha^2} H, \quad 0 \leq t \leq T$$

and if $\alpha < -1$, we have

$$P(t) \leq \frac{H}{(1+\alpha)^2}, \quad \int_\Omega \theta^2 dx \leq \frac{b\lambda^{-2}}{(\alpha+1)^2} H, \quad 0 \leq t \leq T.$$

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