

Generalized Ulam-Hyers stability of an AQ-functional equation in quasi-beta-normed spaces

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abstract

In this paper, we introduce and investigate the general solution of a new functional equation

$$f\left(\frac{x+y}{a} + \frac{z+w}{b}\right) + f\left(\frac{x+y}{a} - \frac{z+w}{b}\right) = \frac{1}{a^2} [(1+a)f(x+y) + (1-a)f(-x-y)] \\ + \frac{1}{b^2} [f(z+w) + f(-z-w)]$$

where $a, b \geq 2$ and discuss its Generalized Hyers - Ulam - Rassias stability in Quasi- β -normed spaces.

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1. INTRODUCTION

In 1940, S. M. Ulam [32] , while he was giving a talk before the mathematics club of the University of Wisconsin, he proposed a number of important unsolved problems. One of the problem is the stability of functional equation. In the last five decades the problem was tackled by numerous authors [1,2,6,8,12,18,22,26]. Its solutions via various forms of functional equations like additive, quadratic, cubic and quartic and its mixed forms were discussed.

Ulam's stability problem states as follows:

Let G be a group and let H be a metric group with metric $d(.,.)$. Given $\epsilon > 0$ does there exists a $\delta > 0$ such that if a function $f : G \rightarrow H$ satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, then there exists a homomorphism $a : G \rightarrow H$ with $d(f(x), a(x)) < \epsilon$ for all $x \in G$?

In 1941, D.H. Hyers[12] considered the case of approximately additive mappings $f : E \rightarrow E'$ where E and E' are Banach spaces. He proved the following celebrated theorem.

Theorem 1.1 (25). *Let E, E' be Banach spaces and let $f : E \rightarrow E'$ be a mapping satisfying*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in E$. Then the limit $a(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and $a : E \rightarrow E'$ is the unique additive mapping satisfying

$$\|f(x) - a(x)\| \leq \epsilon$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$ then a is linear.

From the above property, the additive functional equation $f(x+y) = f(x) + f(y)$ has Hyers-Ulam stability on (E, E') or alternatively that it is stable in the sense of Hyers and Ulam. In 1951, T.Aoki [2] generalized the Hyers theorem and later in 1978, Th.M.Rassias [25] proved a generalization of Hyers theorem, which allows the cauchy difference to be unbounded. It states as follows:

Theorem 1.2 (25). *Let E, E' be two Banach spaces and let $\theta \in [0, \infty)$ and $p \in [0, 1)$. If a function $f : E \rightarrow E'$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \theta [\|x\|^p + \|y\|^p]$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \rightarrow E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2-2^p} \|x\|^p$$

for all $x \in E$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E$ then T is linear.

These ideas become a powerful tool for studying the stability of several functional equations and they have been called Hyers-Ulam-Rassias stability, In 1982-84, J.M.Rassias [22] in the above Theorem [25], he replaced the sum by the product of powers of norms, which is given in the following Theorem.

Theorem 1.3 (22). *Let $f : E \rightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon \|x\|^p \|y\|^p \quad (1.1)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $0 \leq p < \frac{1}{2}$. Then the limit

$$L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \rightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{\epsilon}{2-2^{2p}} \|x\|^{2p} \quad (1.2)$$

for all $x \in E$. If $p < 0$, then the inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. If $p > \frac{1}{2}$ the inequality (1.1) holds for $x, y \in E$ and the limit

$$A(x) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{x}{2^n}\right)$$

exists for all $x \in E$ and $A : E \rightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - A(x)\| \leq \frac{\epsilon}{2^{2p}-2} \|x\|^{2p}$$

for all $x \in E$. If in addition $f : E \rightarrow E'$ is a mapping such that the transformation $t \rightarrow f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is \mathbb{R} -linear mapping.

In 1983, Skof proved Hyers-Ulam-Rassias stability problem for quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y). \quad (1.3)$$

for a class of functions $f : A \rightarrow B$, where A is a normed space and B is a Banach space (see [2][14]). Many results are available on various quadratic functional equations, one can see ([5][7][15][17] [19]). S.M.Jung [15] investigated the Hyers-Ulam-Rassias stability of the quadratic functional equation on pexider type

$$f_1(x+y) + f_2(x-y) = 2f_3(x) + 2f_4(y).$$

The generalized Hyers-Ulam-Rassias stability of a quadratic equation

$$f(x+y+z) + f(x-y) + f(y-z) + f(z-x) = 3f(x) + 3f(y) + 3f(z)$$

was discussed by B.H.Bae and K.W.Kim [3]. In 2005, K.W.Jun and H.M.Kim [18] obtained the general solution of a generalized quadratic and additive type functional equation of the form

$$f(x+ay) + af(x-y) = f(x-ay) + af(x+y)$$

for any integer a with $a \neq -1, 0, 1$. J.M.Rassias [?, 24] derived the stability of the generalized version of the above quadratic equation

$$Q(a_1x_1 + a_2x_2) + Qf(a_2x_1 - a_1x_2) = (a_1^2 + a_2^2) [Q(x_1) + Q(x_2)]$$

which covers a wide range of quadratic functional equations in two variables. Recently, K.Ravi and R.Kodandan [29] discussed the stability of Additive and Quadratic functional equation

$$f\left(\frac{xz}{y} + \frac{yw}{x}\right) + f\left(\frac{xz}{y} - \frac{yw}{x}\right) = 2f\left(\frac{xz}{y}\right) + f\left(\frac{yw}{x}\right) + f\left(-\frac{yw}{x}\right)$$

where $x, y \neq 0$, in non-Archimedean spaces.

In this paper, we introduce and investigate the general solution of a new functional equation

$$f\left(\frac{x+y}{a} + \frac{z+w}{b}\right) + f\left(\frac{x+y}{a} - \frac{z+w}{b}\right) = \frac{1}{a^2} [(1+a)f(x+y) + (1-a)f(-x-y)] \\ + \frac{1}{b^2} [f(z+w) + f(-z-w)] \quad (1.4)$$

and discuss its Generalized Hyers-Ulam-Rassias stability of this equation in quasi- β -Normed spaces. It may be noted that $f(x) = ax^2 + bx + c$ is a solution of the functional equation (1.4)

Before giving the main results, we will present here some basic facts concerning quasi- β -Normed spaces and some preliminary results. We fix a real number β with $0 < \beta \leq 1$ and let \mathbb{K} denote either \mathbb{R} or \mathbb{C} . Let X be a linear space over \mathbb{K} . A quasi- β -norm $\|\cdot\|$ is a real-valued function on X satisfying the following: Let X be a linear space. A quasi-norm $\|\cdot\|$ is real-valued function on X satisfying the following:

- (i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
- (ii) $\|\lambda x\| = |\lambda|^\beta \cdot \|x\|$ for all $\lambda \in \mathbb{K}$ and all $x \in X$.
- (iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$. The pair $(X, \|\cdot\|)$ is called quasi- β -normed space if $\|\cdot\|$ is a quasi- β -norm on X . The smallest possible K is called the modulus of concavity of $\|\cdot\|$. A quasi- β -Banach space is a complete quasi- β -normed space. A quasi- β -norm $\|\cdot\|$ is called a (β, p) -norm ($0 < p \leq 1$) if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi- β -Banach space is called a (β, p) -Banach space. We can refer to [3,30] for the concept of quasi-normed spaces and p -Banach space. Given a p -norm, the formula $d(x, y) = \|x+y\|^p$ gives us a translation invariant metric on X . By the Aoki-Rolewicz theorem [30] (see also [3]), each quasi-norm is equivalent to some p -norm, since it is much easier to work with p -norms than quasi-norms. henceforth we restrict our attention mainly to p -norms. In [31], J.Tabor has investigated a version of the Hyers-Rassias-Gajda theorem (see[8]) in quasi-Banach spaces. We recall that a subadditive function is a function $\phi : E_1 \rightarrow E_2$, having a domain E_1 and a codomain (E_2, \leq) that are both closed under additive, with the following

Replacing x by 0 in (2.3), we arrive that

$$f\left(\frac{y}{a}\right) = \frac{1}{a^2}f(y), \quad \forall y \in E_1. \quad (2.4)$$

Again replacing y by ax in (2.4), we obtain

$$f(ax) = a^2f(x), \quad \forall x \in E_1. \quad (2.5)$$

Replacing $[(x, y), (z, w)]$ by $[(ax, ay), (bz, bw)]$ in (2.2) and using equation (2.5), we obtain

$$f[(x+y) + (z+w)] + f[(x+y) - (z+w)] = 2f(x+y) + 2f(z+w). \quad (2.6)$$

Replacing $(x+y, z+w)$ by (u, v) in (2.6), we obtain

$$f(u+v) + f(u-v) = 2f(u) + 2f(v). \quad (2.7)$$

Again replacing (u, v) by (x, y) in (2.7), we obtain

$$f(x+y) + f(x-y) = 2f(x) + 2f(y), \quad \forall x, y \in E_1.$$

Therefore $f : E_1 \rightarrow E_2$ is quadratic. \square

Theorem 2.2. *If $f : E_1 \rightarrow E_2$ be an odd function, satisfying (1.4) for all $x, y \in E_1$. Then f is additive.*

Proof. Using oddness of f and using (2.1) in (1.4), we obtain

$$f\left(\frac{x+y}{a} + \frac{z+w}{b}\right) + f\left(\frac{x+y}{a} - \frac{z+w}{b}\right) = \frac{2}{a}f(x+y). \quad (2.8)$$

Replacing (z, w) by $(0, 0)$ and using (2.1) in (2.7), we obtain

$$f\left(\frac{x+y}{a}\right) = \frac{1}{a}f(x+y), \quad \forall x, y \in E_1. \quad (2.9)$$

Replacing x by y in (2.8), we obtain

$$f\left(\frac{2y}{a}\right) = \frac{1}{a}f(2y), \quad \forall y \in E_1. \quad (2.10)$$

Replacing $2y$ by ax in (2.10), we arrive

$$f(ax) = af(x), \quad \forall x \in E_1. \quad (2.11)$$

Theorem 3.1. *Assume that there exists a mapping $\phi : E_1 \times E_1 \times E_1 \times E_1 \rightarrow [0, \infty)$ for which an odd mapping $f : E_1 \rightarrow E_2$ satisfies the inequality*

$$\|D_1 f(x, y, z, w)\|_{E_2} \leq \phi(x, y, z, w) \tag{3.1}$$

for all $x, y, z, w \in E_1$, and that the map ϕ is contractively subadditive with a constant L satisfying $a^{1-\beta}L < 1$. Then there exists a unique additive mapping $A : E_1 \rightarrow E_2$ which satisfies (1.4) and the inequality

$$\|f(x) - A(x)\|_{E_2} \leq \left(\frac{a}{2}\right)^\beta \frac{\phi(x, 0, 0, 0)}{\sqrt[p]{\left(\frac{a^{\beta-1}}{L}\right)^p - 1}} \tag{3.2}$$

for all $x \in E_1$.

Proof. Using oddness and (2.1) in (3.1), we obtain

$$\left\| f\left(\frac{x+y}{a} + \frac{z+w}{b}\right) + f\left(\frac{x+y}{a} - \frac{z+w}{b}\right) - \frac{2}{a}f(x+y) \right\|_{E_2} \leq \phi(x, y, z, w). \tag{3.3}$$

For all $x, y, z, w \in E_1$. Replace (y, z, w) by $(0, 0, 0)$ in (3.3), we obtain

$$\left\| 2f\left(\frac{x}{a}\right) - \frac{2}{a}f(x) \right\|_{E_2} \leq \phi(x, 0, 0, 0), \quad \forall x \in E_1. \tag{3.4}$$

Again replacing x by ax in (3.4) and simplifying, we get

$$\left\| f(x) - \frac{1}{a}f(ax) \right\|_{E_2} \leq \frac{1}{2^\beta} \phi(ax, 0, 0, 0) \tag{3.5}$$

for all $x \in E_1$. Therefore it follows from in (3.5) that when we replace $a^i x$ in the place of x and by iterative method

$$\begin{aligned} \left\| \frac{f(a^l x)}{a^l} - \frac{f(a^m x)}{a^m} \right\|_{E_2}^p &\leq \sum_{i=l}^{m-1} \frac{(aL)^p}{2^{\beta p} a^{\beta p i}} \left\| f(a^i x) - \frac{f(a^{i+1} x)}{a} \right\|_{E_2}^p \\ &\leq \frac{(aL)^p}{2^{\beta p}} \sum_{i=l}^{m-1} \frac{1}{a^{\beta p i}} \phi(a^i x, 0, 0, 0)^p \\ &\leq \frac{(aL)^p}{2^{\beta p}} \sum_{i=l}^{m-1} \frac{(aL)^{p i}}{a^{\beta p i}} \phi(x, 0, 0, 0)^p \\ &\leq \frac{(aL)^p}{2^{\beta p}} \phi(x, 0, 0, 0)^p \sum_{i=l}^{m-1} (a^{1-\beta} L)^{p i}. \end{aligned} \tag{3.6}$$

for all $x \in E_1$, which completes the proof of uniqueness. \square

Theorem 3.2. *Assume that there exists a mapping $\phi : E_1 \times E_1 \times E_1 \times E_1 \rightarrow [0, \infty)$ for which an odd mapping $f : E_1 \rightarrow E_2$ satisfies the inequality*

$$\|D_1 f(x, y, z, w)\|_{E_2} \leq \phi(x, y, z, w) \tag{3.7}$$

for all $x, y, z, w \in E_1$, and that the map ϕ is expansively superadditive with a constant L satisfying $a^{\beta-1}L < 1$. Then there exists a unique mapping $A : E_1 \rightarrow E_2$ which satisfies (1.4) and the inequality

$$\|f(x) - A(x)\|_{E_2} \leq \left(\frac{a}{2}\right)^\beta \frac{\phi(x, 0, 0, 0)}{\sqrt[p]{1 - (a^{\beta-1}L)^p}} \tag{3.8}$$

for all $x \in E_1$.

Proof. From (3.4), we obtain

$$\left\|f(x) - af\left(\frac{x}{a}\right)\right\| \leq \left(\frac{a}{2}\right)^\beta \phi(x, 0, 0, 0) \tag{3.9}$$

it follows from (3.9) with $\frac{x}{a^i}$ in place of x and iterative method that

$$\begin{aligned} \left\|a^l f\left(\frac{x}{a^l}\right) - a^m f\left(\frac{x}{a^m}\right)\right\|_{E_2}^p &\leq \sum_{i=l}^{m-1} a^{\beta pi} \left\|f\left(\frac{x}{a^i}\right) - af\left(\frac{x}{a^{i+1}}\right)\right\|_{E_2}^p \\ &\leq \left(\frac{a}{2}\right)^{\beta p} \sum_{i=l}^{m-1} a^{\beta pi} \phi\left(\frac{x}{a^i}, 0, 0, 0\right)^p \\ &\leq \left(\frac{a}{2}\right)^{\beta p} \phi(x, 0, 0, 0) \sum_{i=l}^{m-1} (a^{\beta-1}L)^{pi} \end{aligned} \tag{3.10}$$

for all $x \in E_1$ and for any $m > l \geq 0$. Therefore we see that a mapping $A : E_1 \rightarrow E_2$ defined by

$$A(x) = \lim_{m \rightarrow \infty} a^m f\left(\frac{x}{a^m}\right)$$

is well defined for all $x \in E_1$. Taking the limit $m \rightarrow \infty$ in (3.10) with $l = 0$, we find that the mapping A satisfying the inequality (3.8) near the approximate mapping $f : E_1 \rightarrow E_2$ of (1.4). The remaining proof is similar to that of Theorem 3.1. \square

for all $x, y, z, w \in E_1$. If a mapping $\phi : E_1 \times E_1 \times E_1 \times E_1 \rightarrow [0, \infty)$ satisfies

$$\Phi(x, 0, 0, 0) = \sum_{i=0}^{\infty} (a^\beta K)^i \phi\left(\frac{x}{a^i}, 0, 0, 0\right) < \infty \text{ and } \lim_{m \rightarrow \infty} (a^\beta K)^m \phi\left(\frac{x}{a^m}, 0, 0, 0\right) = 0.$$

for all $x, y, z, w \in E_1$. Then there exists a unique additive mapping $A : E_1 \rightarrow E_2$ such that A satisfies (1.4) and the inequality

$$\|f(x) - A(x)\|_{E_2} \leq K \left(\frac{a}{2}\right)^\beta \Phi(x, 0, 0, 0),$$

for all $x \in E_1$.

Proof. It follows from (3.9) with $\frac{x}{a^i}$ and the similar method to (3.11) that

$$\begin{aligned} \left\| f(x) - a^m f\left(\frac{x}{a^m}\right) \right\|_{E_2} &\leq K \left(\frac{a}{2}\right)^\beta \sum_{i=0}^{m-2} (a^m K)^i \phi\left(\frac{x}{a^i}, 0, 0, 0\right) \\ &\quad + \left(\frac{a}{2}\right)^\beta (a^\beta K)^{m-1} \phi\left(\frac{x}{a^{m-1}}, 0, 0, 0\right) \end{aligned}$$

for all $x \in E_1$ and for any $m > 1$. Therefore we see that a mapping $A : E_1 \rightarrow E_2$ defined by $A(x) = \lim_{m \rightarrow \infty} a^m f\left(\frac{x}{a^m}\right)$ is well defined for all $x \in E_1$. The remaining proof is similar to that of Theorem 3.3. \square

Corollary 3.5. *Let E_1 be a quasi- α -normed linear space with quasi- α -norm $\| \cdot \|$. if there exists a fixed real number $r \in \mathbb{R}$ such that an odd mapping $f : E_1 \rightarrow E_2$ satisfies the functional inequality*

$$\|D_1 f(x, y, z, w)\|_{E_2} \leq \theta (\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$$

for all $x, y, z, w \in E_1$ ($E_1 \setminus \{0\}$ if $r \leq 0$), then there exists a unique additive mapping $A : E_1 \rightarrow E_2$ which satisfies Eq.(1.4) and the inequality

$$\|f(x) - A(x)\| \leq \begin{cases} \frac{K\theta}{2^\beta} \frac{a^{\alpha r}}{1 - K a^{\alpha r - \beta}} & \text{if } K a^{\alpha r} < a^\beta, \\ \left(\frac{K\theta a^\beta}{2^\beta}\right) \frac{1}{1 - K a^{\beta - \alpha r}} & \text{if } K a^\beta < a^{\alpha r}, \end{cases}$$

for all $x \in E_1$ ($E_1 \setminus \{0\}$ if $r \leq 0$).

Proof. By replacing $\phi(x, y, z, w)$ by $(\|x\|^r + \|y\|^r + \|z\|^r + \|w\|^r)$ in Theorem 3.1 and Theorem 3.2, we obtain above result. \square

for all $x \in E_1$ and for any $m > l \geq 0$. Thus it follows that a sequence $\left\{ \frac{f(a^m x)}{a^{2m}} \right\}$ is a cauchy sequence in E_2 and so it converges. Therefore we see that a mapping $A : E_1 \rightarrow E_2$ defined by $Q(x) = \lim_{m \rightarrow \infty} \frac{f(a^m x)}{a^{2m}}$ is well defined for all $x \in E_1$. In addition it is clear from (3.12) that the following inequality

$$\begin{aligned} \|D_1 Q(x, y, z, w)\|_{E_2}^p &= \lim_{m \rightarrow \infty} \frac{\|D_1 f(a^m x, a^m y, a^m z, a^m w)\|_{E_2}^p}{a^{2\beta pm}} \\ &\leq \lim_{m \rightarrow \infty} \frac{\|\varphi(a^m x, a^m y, a^m z, a^m w)\|_{E_2}^p}{a^{2\beta pm}} \\ &\leq \lim_{m \rightarrow \infty} (a^{1-2\beta} L)^{\beta pm} \varphi(x, y, z, w)^p = 0 \end{aligned}$$

holds for all $x, y, z, w \in E_1$ and so the mapping Q is quadratic. Taking the limit $m \rightarrow \infty$ in (3.17) with $l = 0$, we find that

$$\begin{aligned} \|f(x) - Q(x)\|_{E_2}^p &\leq \left(\frac{aL}{2^\beta}\right)^p \varphi(x, 0, 0, 0)^p \sum_{i=0}^{\infty} (a^{1-2\beta} L)^{pi} \\ &\leq \left(\frac{aL}{2^\beta}\right)^p \varphi(x, 0, 0, 0)^p \frac{1}{1 - (a^{1-2\beta} L)^p} \end{aligned}$$

therefore, we get

$$\|f(x) - Q(x)\|_{E_2} \leq \left(\frac{a}{2}\right)^\beta \frac{\varphi(x, 0, 0, 0)}{\sqrt[p]{\left(\frac{a^{2\beta-1}}{L}\right)^p - 1}}.$$

To prove uniqueness, we assume now that there is another function $Q' : E_1 \rightarrow E_2$ which satisfies (1.4) and the inequality (3.13) then it follows that $Q'(ax) = aQ'(x)$, $Q'(a^m x) = a^m Q'(x)$ for all $x \in E_1$ and all $m \in N$. Thus

$$\begin{aligned} \left\| \frac{f(a^m x)}{a^m} \right\|_{E_2} &= \frac{1}{a^{2\beta m}} \left\| f(a^m x) - Q'(a^m x) \right\|_{E_2} \\ &\leq \left(\frac{aL}{a^{2\beta m}}\right)^m \left(\frac{a^2}{2}\right)^\beta \frac{\varphi(a^m x, 0, 0, 0)}{\sqrt[p]{a^{\beta p} - (aL)^p}} \\ &\leq (aL) \left(\frac{a}{2}\right)^\beta (a^{1-2\beta} L)^m \frac{\varphi(x, 0, 0, 0)}{\sqrt[p]{a^{2\beta p} - (aL)^p}} \end{aligned}$$

for all $x \in E_1$ and all $m \in N$. Allow $m \rightarrow \infty$, we get

$$\|Q(x) - Q'(x)\| = 0$$

