Mathematica Aeterna, Vol. 1, 2011, no. 05, 279 - 295

Composition-operator-preserving maps

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Abstract

For a given space X of holomorphic functions in the open unit disc, we determine which self-maps Φ of $\mathcal{L}(\mathsf{X})$ preserve the family $\mathcal{F}_C(\mathsf{X})$ of composition operators leaving X invariant. We show that their surjective multiplicative restrictions to $\mathcal{F}_C(\mathsf{X})$ are exactly of the form $\Phi(T) = A^{-1}TA$ with A a bijective member of $\mathcal{F}_C(\mathsf{X})$. We characterize the norm-preserving ones by the same form with A induced by a rotation. We generalize these results to the semi-multiplicative maps.

Mathematics Subject Classification: Primary 46J15; Secondary 47B33

Keywords: Holomorphic functions, evaluation functionals, composition operators, preservers, (semi-)multiplicative maps, (semi-)anti-multiplicative maps.

Let $\mathsf{H}(\mathsf{D})$ be the algebra of *holomorphic* functions in the open unit disc D and $\mathsf{H}(\mathsf{D},\mathsf{D})$ its subset consisting of all self-maps of D. For a given $\varphi \in \mathsf{H}(\mathsf{D},\mathsf{D})$, the *composition operator* with *symbol* φ is the linear map C_φ sending every $f \in \mathsf{H}(\mathsf{D})$ into $f \circ \varphi : z \longmapsto f\big(\varphi(z)\big)$. In the special case where φ takes only one value $z \in \mathsf{D}$, the operator C_φ is simply the *evaluation functional* δ_z at the point z. For any subspace X of $\mathsf{H}(\mathsf{D})$, we denote by X^* the algebraic dual space of X and by $\mathcal{F}_e(\mathsf{X})$ the subset of X^* consisting of all evaluation functionals. On the other hand, we denote by $\mathcal{L}(\mathsf{X})$ the space of all operators from X into itself (not necessarily continuous) and by $\mathcal{F}_C(\mathsf{X})$ the family of all composition operators leaving X invariant.

The diversity of spaces X has generated a lot of works, many of them relate operator-theoretic properties of C_φ to function-theoretic ones of φ (see e.g. [2] and [9]). As far as we are concerned, we give a new direction by rather considering the whole set $\mathcal{F}_C(\mathsf{X})$ and looking for the self-maps of $\mathcal{L}(\mathsf{X})$ that leave it invariant. Such maps will be called $\mathcal{F}_C(\mathsf{X})$-*preservers*. By this contribution, we intend to provide an extension to our recent work [6], where we have worked out the same problem for the set $\mathcal{F}_e(\mathsf{X})$ which can be considered as a part of $\mathcal{F}_C(\mathsf{X})$. Another source of motivation to our work is the striking similarity between the main results on specific linear preservers given in [5], [14] and second author's recent works (see [10, 11, 12, 13]). In this work, we provide a non-linear version of the problem, due to the non-linear structure of the set $\mathcal{F}_C(\mathsf{X})$. Actually, the stability of this family, under composition, makes the problem treatable with maps either preserving or reversing the order of composition. The first ones will be said *multiplicative* and the last ones *anti-multiplicative.*
In Section 2, we give a sufficient condition for any map to preserve $\mathcal{F}_C(\mathsf{X})$. For multiplicative or anti-multiplicative maps, we provide a complete characterization relying on our basic result in [6]. In Section 3 dealing with surjective multiplicative maps, we obtain an analogous version, for the class $\mathcal{F}_C(\mathsf{X})$, of the main theorems given in [5] and [14]. In Section 4, we determine the multiplicative preservers of $\mathcal{F}_C(\mathsf{X})$ that preserve the norm (respectively, the reduced minimum modulus). In each of the last two sections, we generalize the basic results to the *semi-multiplicative* maps, defined by perturbation of the multiplicative ones with a member of $\mathcal{F}_C(\mathsf{X})$.
Throughout this paper, $\mathbf{p}_0$ and $\mathbf{p}_1$ denote respectively the functions $z \longmapsto 1$ and $z \longmapsto z$. In addition, X is supposed to satisfy, as several well-known spaces, the following conditions in which $\mathbf{p}_0$ denotes the constant function $z \longmapsto 1$ and $\mathbf{p}_1$ the identity function $z \longmapsto z$.

① X is invariant under the multiplication by $\mathbf{p}_1$ and the maps T_a $(a \in \mathsf{D})$: $f \longmapsto f_a$ where $f_a(z) = \frac{f(z)-f(a)}{z-a}$ if $z \in \mathsf{D}\backslash\{a\}$ and $f_a(a) = f'(a)$.

② For all $a \in \mathbb{C}\backslash\mathsf{D}$, X contains an N^{th} root $(N \geq 1)$ of $\frac{1}{\mathbf{p}_1 - a\mathbf{p}_0}$.

③ X contains the space H^∞ of all bounded holomorphic functions in D.

As common examples of such spaces, one can think about the entire space $\mathsf{H}(\mathsf{D})$, Hardy spaces and weighted Bergman spaces. For details, see [6].

1 Composition-operator-preserving maps

The next lemma provides a linear equation in $\mathcal{L}(\mathsf{X})$ which solutions, if they exist, are necessarily in $\mathcal{F}_C(\mathsf{X})$.

Lemma 1.1 *Let C_φ, $C_\psi \in \mathcal{F}_C(\mathsf{X})$ such that φ is not identically constant. If $T \in \mathcal{L}(\mathsf{X})$ satisfies $C_\varphi T = C_\psi$, then T is a composition operator.*

Proof. According to the characterization of $\mathcal{F}_C(\mathsf{X})$ we have given in [6, Theorem 2.3], it would be sufficient to check for T the multiplicativity property. For all $\omega := \varphi(z) \in \Omega := \varphi(\mathsf{D})$, it follows

$$(T\mathbf{p}_0)(\omega) = \big(C_\varphi(T\mathbf{p}_0)\big)(z) = (C_\psi \mathbf{p}_0)(z) = 1.$$

As $T\mathbf{p}_0$ is holomorphic in D and Ω is open, one gets $T\mathbf{p}_0 = \mathbf{p}_0$. On the other hand, for all $f, g \in \mathsf{X}$ such that $fg \in \mathsf{X}$, the holomorphic functions $T(fg)$ and $(Tf)(Tg)$ agree on Ω (and then on D). Indeed, for all $\omega = \varphi(z) \in \Omega$, one has

$$T(fg)(\omega) = [C_\varphi T]\,(fg)(z) = [C_\varphi T]\,f(z)\ \ [C_\varphi T]\,g(z) = (Tf)(Tg)(\omega).$$

The conclusion follows from Theorem 2.3 in [6]. □

Remarks.
❶ In the previous lemma, if the identity holds with $\varphi \equiv a \in \mathsf{D}$, then ψ is constant and $T\mathbf{p}_0(a) = 1$. But T does not necessarily fix $\mathbf{p}_0$.

❷ The conclusion of Lemma 1.1 fails to occur if we exchange C_φ and T in the given identity.

The following theorem gives a sufficient condition for a map $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ to preserve both $\mathcal{F}_C(\mathsf{X})$ and $\mathcal{F}_e(\mathsf{X})$.

Theorem 1.2 *Let $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ and $\varphi \in \mathsf{H}(\mathsf{D}, \mathsf{D})$, non-constant such that $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$. Assume that either*

(i) $$C_\varphi \Phi(T) = C_\varphi T, \quad \textit{for all } T \in \mathcal{F}_C(\mathsf{X});$$

or

(ii) $$C_\varphi \Phi(T) = T C_\varphi, \quad \textit{for all } T \in \mathcal{F}_C(\mathsf{X}).$$

We have the following.

(1) $\Phi(\mathcal{F}_C(\mathsf{X})) \subseteq \mathcal{F}_C(\mathsf{X})$.

(2) $\Phi(\mathcal{F}_e(\mathsf{X})) \subseteq \mathcal{F}_e(\mathsf{X})$. *More precisely, for all* $z \in \mathsf{D}$, *one has*

$$\Phi(\delta_z) = \begin{cases} \delta_z & \textit{if } (i) \textit{ holds} \\ \delta_{\varphi(z)} & \textit{if } (ii) \textit{ holds.} \end{cases}$$

Proof. (1) In both identities (i) and (ii), φ is not constant and the second member is in $\mathcal{F}_C(\mathsf{X})$. So, by Lemma 1.1, $\Phi(T)$ is necessarily in $\mathcal{F}_C(\mathsf{X})$ whenever $T \in \mathcal{F}_C(\mathsf{X})$.
(2) For all $z \in \mathsf{D}$, set $C_\beta = \Phi(\delta_z)$. Under the assumption of (i), one has

$$C_{\beta\circ\varphi} = C_\varphi C_\beta = C_\varphi \Phi(\delta_z) = C_\varphi \delta_z = \delta_z.$$

Thus, β agrees with the constant function $z\mathsf{p}_0$ on the open set $\varphi(\mathsf{D})$ and then on D. This means that $\Phi(\delta_z) = C_\beta = \delta_z$.
Assume now that (ii) holds. One can write

$$C_{\beta\circ\varphi} = C_\varphi \Phi(\delta_z) = \delta_z C_\varphi = \delta_{\varphi(z)}.$$

By the same argument as before, this gives $\Phi(\delta_z) = C_\beta = \delta_{\varphi(z)}$, and we are done. □

Remark.
According to the second remark following Lemma 1.1, exchanging C_φ and $\Phi(T)$ in (i) and (ii) cannot ensure the conclusion of Theorem 1.2.

In the sequel, we say that $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ is *multiplicative* (*anti-multiplicative*) on $\mathcal{F}_C(\mathsf{X})$ if $\Phi(ST) = \Phi(S)\Phi(T)$ (respectively, $\Phi(ST) = \Phi(T)\Phi(S)$) for all $S, T \in \mathcal{F}_C(\mathsf{X})$.

Theorem 1.3 *Let* $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ *be either multiplicative or anti-multiplicative on* $\mathcal{F}_C(\mathsf{X})$ *such that* $\Phi_{|\mathcal{F}_C(\mathsf{X})}$ *is not constant. If* $\Phi(\mathcal{F}_e(\mathsf{X})) \subseteq \mathcal{F}_C(\mathsf{X})$ *then*

$$\Phi(\mathcal{F}_e(\mathsf{X})) \subseteq \mathcal{F}_e(\mathsf{X}).$$

Proof. Given any $z \in \mathsf{D}$, we denote by β the symbol of the composition operator $\Phi(\delta_z)$. We are going to show that β is constant, which means that $\Phi(\delta_z) \in \mathcal{F}_e(\mathsf{X})$. We have

$$C_\beta = \Phi(\delta_z) = \Phi(\delta_z\delta_z) = \Phi(\delta_z)\Phi(\delta_z) = C_\beta C_\beta = C_{\beta\circ\beta}.$$

So, $\beta = \beta \circ \beta$. If β were not constant, then $\Omega := \beta(\mathsf{D})$ would be open and for all $\omega \in \Omega$, $\beta(\omega) - \omega = 0$. As $\beta - \mathsf{p}_1$ is analytic in D, the last identity would also hold in D, or equivalently, $\beta = \mathsf{p}_1$. On the other hand, for all $T \in \mathcal{F}_C(\mathsf{X})$, one would have

$$\Phi(T) = \begin{cases} \Phi(T)\Phi(\delta_z) = \Phi(T\delta_z) & \text{if } \Phi \text{ is multiplicative on } \mathcal{F}_C(\mathsf{X}) \\ \Phi(\delta_z)\Phi(T) = \Phi(T\delta_z) & \text{if } \Phi \text{ is anti-multiplicative on } \mathcal{F}_C(\mathsf{X}). \end{cases}$$

Consequently,

$$\Phi(T) = \Phi(T\delta_z) = \Phi(\delta_z) = C_\beta = I,$$

which would be in contradiction with the hypothesis. Hence, β is necessarily constant. This completes the proof. $\square$

For a given map $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ such that $\Phi(\mathsf{X}^*) \subseteq \mathsf{X}^*$, we introduce the associated operator T_Φ on X by $(T_\Phi f)(z) = \Phi(\delta_z)f$ for all $z \in \mathsf{D}$. Note that T_Φ may not send X into itself. When Φ is only defined on $\mathcal{F}_C(\mathsf{X})$ such that $\Phi\big(\mathcal{F}_e(\mathsf{X})\big) \subseteq \mathsf{X}^*$, we also denote by T_Φ the operator $T_{\widetilde{\Phi}}$, where $\widetilde{\Phi}$ is the trivial extension of Φ to $\mathcal{L}(\mathsf{X})$ taking $\mathcal{L}(\mathsf{X})\backslash\mathcal{F}_C(\mathsf{X})$ into $\{0\}$. In the sequel, any self-map Φ of $\mathcal{L}(\mathsf{X})$ $\big($resp. $\mathcal{F}_C(\mathsf{X})\big)$ is assumed to satisfy the following stability condition : $\Phi(\mathsf{X}^*) \subseteq \mathsf{X}^*$ and $T_\Phi(\mathsf{X}) \subseteq \mathsf{X}$ $\big($resp. $T_\Phi(\mathsf{X}) \subseteq \mathsf{X}\big)$. This will be needed to apply our result [6, Theorem 3.3] describing all the self-maps of X^* preserving $\mathcal{F}_e(\mathsf{X})$.

In the following theorem, we give a necessary condition for a self-map of $\mathcal{L}(\mathsf{X})$ to preserve $\mathcal{F}_C(\mathsf{X})$.

Theorem 1.4 *Let $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ be either multiplicative or anti-multiplicative on $\mathcal{F}_C(\mathsf{X})$ such that $\Phi_{|\mathcal{F}_C(\mathsf{X})}$ is not constant. If $\Phi(\mathcal{F}_C(\mathsf{X})) \subseteq \mathcal{F}_C(\mathsf{X})$ then there is a unique $\varphi \in \mathsf{H}(\mathsf{D},\mathsf{D})$ such that $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$ and*

$$C_\varphi \Phi(T) = TC_\varphi \quad \textit{for all } T \in \mathcal{F}_C(\mathsf{X}).$$

Moreover, if $\Phi_{|\mathcal{F}_e(\mathsf{X})}$ is not constant, then neither is φ.

Proof. Taking Theorem 1.3 into account, our result [6, Theorem 3.3] ensures the existence of a unique $\varphi \in \mathsf{H}(\mathsf{D},\mathsf{D})$ such that $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$ and

$$\Phi(\delta_z) = \delta_{\varphi(z)} \text{ for all } z \in \mathsf{D}.$$

Given any $T \in \mathcal{F}_C(\mathsf{X})$, there exist $\alpha, \beta \in \mathsf{H}(\mathsf{D},\mathsf{D})$ such that $T = C_\alpha$ and $\Phi(T) = C_\beta$. If Φ is multiplicative on $\mathcal{F}_C(\mathsf{X})$, then one has

$$\delta_{\varphi\circ\alpha(z)} = \Phi(\delta_{\alpha(z)}) = \Phi(\delta_z C_\alpha) = \delta_{\varphi(z)} C_\beta = \delta_{\beta\circ\varphi(z)},$$

from which it follows that $\beta \circ \varphi = \varphi \circ \alpha$. This gives

$$C_\varphi \Phi(T) = C_\varphi C_\beta = C_{\beta\circ\varphi} = C_{\varphi\circ\alpha} = C_\alpha C_\varphi = TC_\varphi.$$

Now, if Φ is anti-multiplicative, a similar argument leads to the identity $\beta\circ\varphi = \varphi$ which implies that $C_\varphi\Phi(T) = C_\varphi$ and to the equality $\varphi \circ \alpha = \varphi$ which gives $TC_\varphi = C_\varphi$. Hence, $C_\varphi\Phi(T) = TC_\varphi$. Finally, to get the uniqueness of φ, take any ψ satisfying the same property as φ in Theorem 1.4. In particular, for every $T = \delta_z$, it follows that

$$\delta_{\varphi(z)} = C_\psi \delta_{\varphi(z)} = C_\psi \Phi(\delta_z) = \delta_z C_\psi = \delta_{\psi(z)}.$$

This means $\psi = \varphi$. The end of the statement is due to the action of Φ on $\mathcal{F}_e(\mathsf{X})$, described above. $\square$

Remark.
Actually, the further assumption that Φ is not constant on $\mathcal{F}_e(\mathsf{X})$ provides a more precise information on φ. Indeed, we will see in the next section that φ is necessarily one-to-one.

One can deduce the following corollary immediately from Theorem 1.4 and Theorem 1.2.

Corollary 1.5 *Let $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ be multiplicative on $\mathcal{F}_C(\mathsf{X})$ such that $\Phi_{|\mathcal{F}_e(\mathsf{X})}$ is not constant. The following are equivalent.*

(1) $\Phi(\mathcal{F}_C(\mathsf{X})) \subseteq \mathcal{F}_C(\mathsf{X})$;

(2) *there is a unique $\varphi \in \mathsf{H}(\mathsf{D}, \mathsf{D})$, non-constant such that $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$ and*

$$C_\varphi \Phi(T) = TC_\varphi, \quad \textit{for all } T \in \mathcal{F}_C(\mathsf{X}).$$

Corollary 1.6 *Let $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ be anti-multiplicative on $\mathcal{F}_C(\mathsf{X})$ such that $\Phi_{|\mathcal{F}_C(\mathsf{X})}$ is not constant. The following are equivalent.*

(1) $\Phi(\mathcal{F}_C(\mathsf{X})) \subseteq \mathcal{F}_C(\mathsf{X})$;

(2) *there is a unique $a \in \mathsf{D}$ such that*

$$\Phi(\mathcal{F}_C(\mathsf{X})) \subseteq \{C_\beta \in \mathcal{F}_C(\mathsf{X});\ \beta(a) = a\}.$$

Proof. We only need to show the direction '(1) $\Longrightarrow$ (2)'. According to the proof of Theorem 1.4, there is a unique $\varphi \in \mathsf{H}(\mathsf{D}, \mathsf{D})$ with $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$ such that $\Phi_{|\mathcal{F}_e(\mathsf{X})} = {C_\varphi}^*{}_{|\mathcal{F}_e(\mathsf{X})}$ and for all $T \in \mathcal{F}_C(\mathsf{X})$, we have

$$(\star)\ \ TC_\varphi = C_\varphi \qquad \text{and} \qquad (\star\star)\ \ C_\varphi \Phi(T) = C_\varphi.$$

Taking $T \in \mathcal{F}_e(\mathsf{X})$ in $(\star)$ forces C_φ to be in $\mathcal{F}_e(\mathsf{X})$ and then φ to be constant. Let a be this constant. For all $z \in \mathsf{D}$, one gets

$$\Phi(\delta_z) = {C_\varphi}^*(\delta_z) = \delta_{\varphi(z)} = \delta_a.$$

On the other hand, by writing $\Phi(T) = C_\beta \in \mathcal{F}_C(\mathsf{X})$ and $\varphi = a\mathsf{p}_0$ in $(\star\star)$, it follows that $\delta_{\beta(a)} = \delta_a$ which gives $\beta(a) = a$. The uniqueness of a follows from the fact that $\Phi(\delta_z) = \delta_a$. This achieves the proof. $\square$

From the previous corollary, one can easily deduce the following.

Corollary 1.7 *There is no surjective anti-multiplicative self-map of $\mathcal{F}_C(\mathsf{X})$.*

2 Surjective multiplicative self-maps of $\mathcal{F}_C(\mathsf{X})$

Next, we are going to set up the main ingredients we need to determine all surjective multiplicative self-maps of $\mathcal{F}_C(\mathsf{X})$. We denote by $\mathsf{Aut}(\mathsf{D})$ the group of the automorphisms of D; i.e. the set of all bijective $\varphi \in \mathsf{H}(\mathsf{D},\mathsf{D})$. For any $r > 0$, the open disk centered at 0 with radius r will be denoted by $\mathsf{D}(0, r)$ and its boundary by $\partial\mathsf{D}(0, r)$.

Lemma 2.1 *Let φ be non constant in $\mathsf{H}(\mathsf{D},\mathsf{D})$. If, for all $\alpha \in \mathsf{Aut}(\mathsf{D})$, there is $\beta \in \mathsf{H}(\mathsf{D},\mathsf{D})$ such that $\beta \circ \varphi = \varphi \circ \alpha$, then φ is injective.*

Proof. First, we claim that

$$\varphi(\mathsf{D}\backslash\{0\}) \subseteq \varphi(\mathsf{D})\backslash\{\varphi(0)\}.$$

Indeed, suppose the contrary; i. e., there exists $u \in \mathsf{D}\backslash\{0\}$ such that $\varphi(u) = \varphi(0)$. Set $r = |u|$ and call v a point at which the continuous function $|\varphi|$ on the closure of $\mathsf{D}(0, \mathsf{r})$ reaches its maximum on the compact subset $\partial\mathsf{D}(0, r)$. Consider the rotation α of D sending u into v. Since $\alpha \in \mathsf{Aut}(\mathsf{D})$, there exists $\beta \in \mathsf{H}(\mathsf{D},\mathsf{D})$ such that $\beta \circ \varphi = \varphi \circ \alpha$. Thus

$$\varphi(0) = \varphi \circ \alpha(0) = \beta \circ \varphi(0) = \beta \circ \varphi(u) = \beta \circ \varphi \circ \alpha^{-1}(v) = \varphi(v).$$

From this, it follows that $\varphi(0) = \sup_{|z|=r} |\varphi(z)|$. Therefore, φ has to be constant, according to the maximum principle. But this is not allowed according to the hypothesis. Hence, the inclusion above is true.
Now, to get the conclusion of this lemma, let $a, b \in \mathsf{D}$ such that $a \neq b$. Consider the Mbius transform φ_a defined by $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$. It is well known that $\varphi_a \in \mathsf{Aut}(\mathsf{D})$ and $\varphi_a^{-1} = \varphi_a$. So, for all $\alpha \in \mathsf{Aut}(\mathsf{D})$, the hypothesis on φ, applied with the automorphism $\varphi_a \circ \alpha \circ \varphi_a$, gives one $\beta \in \mathsf{H}(\mathsf{D},\mathsf{D})$ such that $\beta \circ \varphi = \varphi \circ (\varphi_a \circ \alpha \circ \varphi_a)$. Composing with φ_a at the right of each side, one gets $\beta \circ \psi = \psi \circ \alpha$ where $\psi = \varphi \circ \varphi_a$. As $\varphi_a(b) \neq 0$ and ψ satisfies the hypothesis of this lemma, one can use the inclusion above with ψ instead of φ to get $\psi(0) \neq \psi(\varphi_a(b))$. Therefore, $\varphi(a) \neq \varphi(b)$. This achieves the proof. □

The following result is a consequence of the previous lemma.

Corollary 2.2 *If $\varphi \in \mathsf{H}(\mathsf{D},\mathsf{D})$ is surjective and for all $\alpha \in \mathsf{Aut}(\mathsf{D})$, there is $\beta \in \mathsf{H}(\mathsf{D},\mathsf{D})$ such that $\beta \circ \varphi = \varphi \circ \alpha$, then $\varphi \in \mathsf{Aut}(\mathsf{D})$.*

According to a non-trivial result from [3, Corollary], a bijective member of $\mathcal{F}_C(\mathsf{X})$ is automatically induced by one $\varphi \in \mathsf{Aut}(\mathsf{D})$.

We reach now our target result, describing any surjective multiplicative self-map of $\mathcal{F}_C(\mathsf{X})$, by a similarity property involving a bijective member of this family. Here, X is supposed to be invariant under rotations of D.

Theorem 2.3 *Let $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ be multiplicative. The following are equivalent.*

(1) $\Phi(\mathcal{F}_C(\mathsf{X})) = \mathcal{F}_C(\mathsf{X})$;

(2) $\mathcal{F}_e(\mathsf{X}) \subseteq \Phi(\mathcal{F}_C(\mathsf{X})) \subseteq \mathcal{F}_C(\mathsf{X})$;

(3) *there exist a unique bijection $A \in \mathcal{F}_C(\mathsf{X})$ and $\Psi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ with $\Psi_{|\mathcal{F}_C(\mathsf{X})} \equiv 0$ such that*

$$\Phi(T) = A^{-1}TA + \Psi(T), \quad \textit{for all } T \in \mathcal{L}(\mathsf{X});$$

(4) *there exists a unique $\varphi \in \mathsf{Aut}(\mathsf{D})$ with $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$ such that*

$$\Phi(C_\alpha) = C_{\varphi\circ\alpha\circ\varphi^{-1}},$$

for all $\alpha \in \mathsf{H}(\mathsf{D}, \mathsf{D})$ with $C_\alpha(\mathsf{X}) \subseteq \mathsf{X}$.

Proof. (1) $\Longrightarrow$ (2) and (4) $\Longrightarrow$ (1) are obvious.
(3) $\Longrightarrow$ (4). By [3, Corollary], there exists a unique $\varphi \in \mathsf{Aut}(\mathsf{D})$ with $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$, $C_{\varphi^{-1}}(\mathsf{X}) \subseteq \mathsf{X}$ and $A^{-1} = C_\varphi^{-1} = C_{\varphi^{-1}}$. Taking $T = C_\alpha$ in (3) leads to (4).
(2) $\Longrightarrow$ (3). Φ satisfies the hypothesis of Theorem 1.4 ensuring therefore the existence and uniqueness of one $\varphi \in \mathsf{H}(\mathsf{D}, \mathsf{D})$ with $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$ such that

$$C_\varphi C_\beta = C_\alpha C_\varphi \quad \text{for all } C_\alpha \in \mathcal{F}_C(\mathsf{X}),$$

where $C_\beta = \Phi(C_\alpha)$. Therefore, $C_{\beta\circ\varphi} = C_{\alpha\circ\varphi}$, or equivalently

$$(\star) \qquad \beta \circ \varphi = \varphi \circ \alpha.$$

In particular, for any non-constant α, this forces β to be non-constant too. Indeed, suppose the contrary; i.e. there exist $C_{\alpha_0} \in \mathcal{F}_C(\mathsf{X})$ and $a \in \mathsf{D}$ such that $\beta_0 = a\mathsf{p_0}$, where $C_{\beta_0} = \Phi(C_{\alpha_0})$. Then, from $(\star)$ applied with α_0 and β_0, it follows that φ coincides with $a\mathsf{p_0}$ on the open set $\alpha_0(\mathsf{D})$ and then on D. Therefore, thanks to $(\star)$, Φ sends each member of $\mathcal{F}_C(\mathsf{X})$ into a member with symbol fixing a. But this makes a contradiction with (2). Hence, Φ leaves $\mathcal{F}_C(\mathsf{X})\backslash\mathcal{F}_e(\mathsf{X})$ invariant. Taking (2) into account, this gives $\mathcal{F}_e(\mathsf{X}) \subseteq \Phi(\mathcal{F}_e(\mathsf{X}))$. As we also have the reverse inclusion by Theorem 1.3, we get $\Phi(\mathcal{F}_e(\mathsf{X})) = \mathcal{F}_e(\mathsf{X})$ and this, together with $(\star)$, leads to the surjectivity of φ. Now that φ satisfies the hypothesis of Corollary 2.2, we deduce that $\varphi \in \mathsf{Aut}(\mathsf{D})$ and then $A = C_\varphi$ is a bijective member of $\mathcal{F}_C(\mathsf{X})$, according to [3, Corollary]. One can therefore define the map Ψ on $\mathcal{L}(\mathsf{X})$ by

$$\Psi(T) = \Phi(T) - A^{-1}TA.$$

Clearly, for all $T \in \mathcal{F}_C(\mathsf{X})$, one has $A\Psi(T) \equiv 0$, or equivalently $\Psi(T) \equiv 0$. Thus, Φ has the form given in (3). Finally, by restricting Φ to $\mathcal{F}_C(\mathsf{X})$, one can deduce the uniqueness of A from that of φ. $\square$

From the previous theorem and by applying Theorem 1.3 to $\widetilde{\Phi}$, one can easily derive the following, in which the assumption that Φ is not constant will be useful to ensure the stability of X^* under $\widetilde{\Phi}$.

Corollary 2.4 *Let $\Phi : \mathcal{F}_C(\mathsf{X}) \longrightarrow \mathcal{F}_C(\mathsf{X})$ be multiplicative and non-constant. The following are equivalent.*

(1) Φ *is surjective;*

(2) $\mathcal{F}_e(\mathsf{X}) \subseteq \Phi(\mathcal{F}_C(\mathsf{X}))$;

(3) $\mathcal{F}_e(\mathsf{X}) = \Phi(\mathcal{F}_e(\mathsf{X}))$;

(4) *there exists a unique bijection $A \in \mathcal{F}_C(\mathsf{X})$ such that*

$$\Phi(T) = A^{-1}TA \quad \textit{for all } T \in \mathcal{F}_C(\mathsf{X});$$

(5) Φ *is bijective.*

As application of this corollary, we will show that the set of all bijective multiplicative self-maps of $\mathcal{F}_C(\mathsf{X})$ can be identified with $\mathsf{Aut}(\mathsf{D})$. Indeed, observe that the set $\mathsf{Mul}(\mathcal{F}_C(\mathsf{X}))$ of all multiplicative self-maps of $\mathcal{F}_C(\mathsf{X})$ is stable under the composition giving therefore a group structure to its subset $\widetilde{\mathsf{Mul}}(\mathcal{F}_C(\mathsf{X}))$, consisting of its bijective members. One can also note that, for any space X as in [3, Corollary], the bijective members of $\mathcal{F}_C(\mathsf{X})$ form a group for the composition. We denote this group by $\widetilde{\mathcal{F}_C}(\mathsf{X})$. Here is a statement of this application where '$\approx$' (resp. '$\asymp$') means 'isomorphic to' (resp. 'anti-isomorphic to').

Corollary 2.5 *Let X be invariant under $\mathsf{Aut}(\mathsf{D})$. We have the following.*

(i) $\mathsf{Aut}(\mathsf{D}) \asymp \widetilde{\mathcal{F}_C}(\mathsf{X})$ *and* $\widetilde{\mathcal{F}_C}(\mathsf{X}) \asymp \widetilde{\mathsf{Mul}}(\mathcal{F}_C(\mathsf{X}))$.

(ii) $\widetilde{\mathsf{Mul}}(\mathcal{F}_C(\mathsf{X})) \approx \mathsf{Aut}(\mathsf{D})$.

Proof. (i) Consider the maps Λ and Θ defined respectively on $\mathsf{Aut}(\mathsf{D})$ and $\widetilde{\mathcal{F}_C}(\mathsf{X})$ by $\Lambda(\varphi) = C_\varphi$ and $\Theta(A) = \Phi_A$ where $\Phi_A(T) = A^{-1}TA$ for all $T \in \mathcal{F}_C(\mathsf{X})$. It is not difficult to see that Λ takes $\mathsf{Aut}(\mathsf{D})$ into $\widetilde{\mathcal{F}_C}(\mathsf{X})$ and Θ sends $\widetilde{\mathcal{F}_C}(\mathsf{X})$ into $\widetilde{\mathsf{Mul}}(\mathcal{F}_C(\mathsf{X}))$. Now, respectively by '(1) $\Longrightarrow$ (2)' of [3, Corollary] and '(4) $\Longrightarrow$ (5)' of Corollary 2.4, one can see that Λ and Θ are respectively bijective. Eventually, since they are anti-homomorphisms, it follows that Λ

and Θ are group anti-isomorphisms.
(ii) This follows immediately from (i). $\square$

By applying Corollary 2.5 with the entire space $\mathsf{H}(\mathsf{D})$, one can easily deduce the following.

Corollary 2.6 *Let* X *be invariant under* $\mathsf{Aut}(\mathsf{D})$*. We have*

$$\widetilde{\mathcal{F}_C}(\mathsf{X}) \approx \widetilde{\mathcal{F}_C}(\mathsf{H}(\mathsf{D})) \quad \textit{and} \quad \widetilde{\mathsf{Mul}}(\mathcal{F}_C(\mathsf{X})) \approx \widetilde{\mathsf{Mul}}(\mathcal{F}_C(\mathsf{H}(\mathsf{D}))).$$

Next, we present another application of Theorem 2.3 giving an extension of it to a family of maps larger than $\mathsf{Mul}(\mathcal{F}_C(\mathsf{X}))$. Given any bijective members A and B in $\mathcal{F}_C(\mathsf{X})$, one can observe that the operator $\Phi_{A,B} : T \longmapsto ATB$ sends $\mathcal{F}_C(\mathsf{X})$ surjectively (in fact bijectively) into itself. On the other hand, note that the map $U\Phi_{A,B} : T \longmapsto U\Phi_{A,B}(T)$, with $U = (AB)^{-1}$, is in $\widetilde{\mathsf{Mul}}(\mathcal{F}_C(\mathsf{X}))$, as it is exactly the map $\Phi_B := \Phi_{B,B}$. It follows therefore that $\Phi_{A,B}$ satisfies the following :

$$\circledast \qquad \exists U \in \mathcal{F}_C(\mathsf{X});\ \forall\, S, T \in \mathcal{F}_C(\mathsf{X}),\ \Phi_{A,B}(ST) = \Phi_{A,B}(S)U\Phi_{A,B}(T).$$

More generally, any map $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ satisfying $\circledast$ instead of $\Phi_{A,B}$ is said to be *semi-multiplicative* on $\mathcal{F}_C(\mathsf{X})$. Remark that any multiplicative map on $\mathcal{F}_C(\mathsf{X})$ is semi-multiplicative. The following result ensures the converse of what has been said above by giving a more general version of Theorem 2.3. Also here, X is supposed to be invariant under rotations of D.

Theorem 2.7 *Let* $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ *be semi-multiplicative on* $\mathcal{F}_C(\mathsf{X})$*. The following are equivalent.*

(1) $\Phi(\mathcal{F}_C(\mathsf{X})) = \mathcal{F}_C(\mathsf{X})$;

(2) $\mathcal{F}_e(\mathsf{X}) \subseteq \Phi(\mathcal{F}_C(\mathsf{X})) \subseteq \mathcal{F}_C(\mathsf{X})$ *and* $\Phi(\mathcal{F}_C(\mathsf{X})) \cap \widetilde{\mathcal{F}_C}(\mathsf{X}) \neq \emptyset$;

(3) *there exists a unique* $(A, B) \in (\widetilde{\mathcal{F}_C}(\mathsf{X}))^2$ *and* $\Psi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ *with* $\Psi_{|_{\mathcal{F}_C(\mathsf{X})}} \equiv 0$ *such that*

$$\Phi(T) = ATB + \Psi(T), \quad \textit{for all } T \in \mathcal{L}(\mathsf{X});$$

(4) *there exists a unique* $(\varphi, \psi) \in (\mathsf{Aut}(\mathsf{D}))^2$ *with* $C_\varphi(\mathsf{X}) \subseteq \mathsf{X}$ *and* $C_\psi(\mathsf{X}) \subseteq \mathsf{X}$ *such that*

$$\Phi(C_\alpha) = C_{\psi\circ\alpha\circ\varphi}$$

for all $\alpha \in \mathsf{H}(\mathsf{D}, \mathsf{D})$ *with* $C_\alpha(\mathsf{X}) \subseteq \mathsf{X}$.

Proof. It is clear that (3) $\Longrightarrow$ (4) $\Longrightarrow$ (1) $\Longrightarrow$ (2).
(2) $\Longrightarrow$ (3). First, remark that if Φ is semi-multiplicative, then there exists $U \in \mathcal{F}_C(\mathsf{X})$ such that

$$(\star) \qquad \Phi(T) = \Phi(T)U\Phi(I) = \Phi(I)U\Phi(T),$$

$$(\star\star) \qquad U\Phi \text{ is multiplicative on } \mathcal{F}_C(\mathsf{X}).$$

It follows from (2) and ($\star$) that

$$U\Phi(I) = \Phi(I)U = I.$$

So $U \in \widetilde{\mathcal{F}_C}(\mathsf{X})$ and this gives (2) of Theorem 2.3 with $U\Phi$ instead of Φ. Taking ($\star\star$) into account, Theorem 2.3 ensures that there exist $B \in \widetilde{\mathcal{F}_C}(\mathsf{X})$ and $\Psi_1 : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ with $\Psi_{1|_{\mathcal{F}_C(\mathsf{X})}} \equiv 0$ such that $U\Phi = \Phi_B + \Psi_1$, or equivalently, $\Phi = U^{-1}\Phi_B + U^{-1}\Psi_1$. Hence, by setting $A = (BU)^{-1}$ which is in $\widetilde{\mathcal{F}_C}(\mathsf{X})$ according to [3, Corollary] and by taking $\Psi = U^{-1}\Psi_1$, one clearly gets the desired form of Φ. For the uniqueness of A and B, one can verify that $B = T_\Phi$ and $A = \Phi(I)B^{-1}$. This achieves the proof. □

From Theorem 2.7, we deduce the following result as analogous version of Corollary 2.4.

Corollary 2.8 *Let $\Phi : \mathcal{F}_C(\mathsf{X}) \longrightarrow \mathcal{F}_C(\mathsf{X})$ be semi-multiplicative and non-constant. The following are equivalent.*

(1) Φ *is surjective;*

(2) $\mathcal{F}_e(\mathsf{X}) \subseteq \Phi(\mathcal{F}_C(\mathsf{X}))$ *and* $\Phi(\mathcal{F}_C(\mathsf{X})) \cap \widetilde{\mathcal{F}_C}(\mathsf{X}) \neq \emptyset$;

(3) $\mathcal{F}_e(\mathsf{X}) = \Phi(\mathcal{F}_e(\mathsf{X}))$ *and* $\Phi(\mathcal{F}_C(\mathsf{X})) \cap \widetilde{\mathcal{F}_C}(\mathsf{X}) \neq \emptyset$;

(4) *there exists a unique* $(A, B) \in (\widetilde{\mathcal{F}_C}(\mathsf{X}))^2$ *such that*

$$\Phi(T) = ATB \quad \textit{for all} \;\; T \in \mathcal{F}_C(\mathsf{X});$$

(5) Φ *is bijective.*

Remarks.
❶ The semi-multiplicative maps leaving $\mathcal{F}_C(\mathsf{X})$ invariant can be characterized by a slight modification of Corollary 1.5. By applying this corollary with the map $U\Phi$, one can easily get the following generalization.

Proposition 2.9 *Let $\Phi : \mathcal{L}(\mathsf{X}) \longrightarrow \mathcal{L}(\mathsf{X})$ be semi-multiplicative on $\mathcal{F}_C(\mathsf{X})$ such that $\Phi_{|\mathcal{F}_e(\mathsf{X})}$ is not constant. The following are equivalent.*

(1) $\Phi(\mathcal{F}_C(\mathsf{X})) \subseteq \mathcal{F}_C(\mathsf{X})$;

(2) *there is a unique* $(A, B) \in (\mathcal{F}_C(\mathsf{X}) \backslash \mathcal{F}_e(\mathsf{X}))^2$ *such that*

$$A\Phi(T) = TB \quad \text{for all } T \in \mathcal{F}_C(\mathsf{X}).$$

❷ From Theorem 2.3 or Corollary 2.4, one can deduce for $\mathcal{F}_C(\mathsf{X})$, an analogous version of the main theorem in each of papers [5] and [14], where the authors have obtained the maps Φ_A (A invertible) as surjective linear self-maps of the space $\mathcal{B}(\mathsf{Y})$ of all bounded operators on a given Banach space Y, which preserve the spectrum in [5] and the invertibility in [14]. In contrast with $\mathcal{B}(\mathsf{Y})$, the set $\mathcal{F}_C(\mathsf{X})$ is not a linear space. That is why we have determined the maps Φ_A among the multiplicative maps rather than the linear ones.

3 Maps preserving the norm of bounded composition operators

In [6], we have characterized the maps that preserve the norm of any bounded δ_z with norm depending injectively on $|z|$. In order to study the same question for the maps acting on $\mathcal{F}_C(\mathsf{X})$, we will consider any normed space X as before such that $\mathcal{F}_C(\mathsf{X}) \subset \mathcal{B}(\mathsf{X})$, where $\mathcal{B}(\mathsf{X})$ denotes the space of all bounded operators on X. In particular, we assume that there exists a one-to-one positive function h on $[0, 1)$ such that $\|\delta_z\| = h(|z|)$ for all $z \in \mathsf{D}$. Moreover, any rotation of D is supposed to induce an isometric composition operator on X. Notice that the Hardy spaces and the Bergman ones satisfy all of these conditions. Here is how one can describe the multiplicative norm-preserving self-maps of $\mathcal{F}_C(\mathsf{X})$.

Theorem 3.1 *Let* $\Phi : \mathcal{F}_C(\mathsf{X}) \longrightarrow \mathcal{F}_C(\mathsf{X})$ *be multiplicative and non-constant. The following are equivalent.*

(1) $\|\Phi(T)\| = \|T\|$, *for all* $T \in \mathcal{F}_C(\mathsf{X})$;

(2) $\|\Phi(\delta_z)\| = \|\delta_z\|$, *for all* $z \in \mathsf{D}$;

(3) *there exists a unique rotation* ρ *of* D *such that*

$$\Phi(T) = C_{\rho^{-1}} T C_\rho, \quad \text{for all } T \in \mathcal{F}_C(\mathsf{X}).$$

Proof. '(1) $\Longrightarrow$ (2)' is obvious and '(3) $\Longrightarrow$ (1)' holds thanks to the rotations property for the space X.
(2) $\Longrightarrow$ (3) From the assumption on the norm of any δ_z, it is clear that Φ is not constant (this is useless, according to the hypothesis). Hence, by Theorems

1.3 and 1.4 applied to the extension $\widetilde{\Phi}$ of Φ (given in Section 3), there is a unique $\varphi \in \mathsf{H}(\mathsf{D}, \mathsf{D})$ with $C_\varphi \in \mathcal{F}_C(\mathsf{X})$ such that

$$\Phi(\delta_z) = \delta_{\varphi(z)} \quad \text{and} \quad C_\varphi \Phi(T) = TC_\varphi,$$

for all $z \in \mathsf{D}$ and $T \in \mathcal{F}_C(\mathsf{X})$. Now, according to Theorem 4.3 in [6] applied with the map ${C_\varphi}^*$ which agrees with Φ on $\mathcal{F}_e(\mathsf{X})$, we deduce that φ is a rotation and we are done. □

From the previous theorem, one can easily derive the following.

Corollary 3.2 *Let* $\Phi : \mathcal{F}_C(\mathsf{X}) \longrightarrow \mathcal{F}_C(\mathsf{X})$ *be multiplicative and non-constant. If* $\|\Phi(\delta_z)\| = \|\delta_z\|$, *for all* $z \in \mathsf{D}$, *then* Φ *is bijective.*

Remark.
There is no anti-multiplicative norm-preserving self-map of $\mathcal{F}_C(\mathsf{X})$. Indeed, if there were such a map, then, according to the proof of Corollary 1.6, Φ would be constant on $\mathcal{F}_e(\mathsf{X})$, giving therefore a contradiction with the assumption made on the norm in this set.

Moving to the the *semi-multiplicative* maps, we need the following result giving a necessary condition for the contractivity of any member of $\mathcal{F}_C(\mathsf{X})$.

Proposition 3.3 *Assume that* h *is strictly increasing. Let* $C_\varphi \in \mathcal{F}_C(\mathsf{X})$. *If* $\|C_\varphi\| \leq 1$, *then* $\varphi(0) = 0$.

Proof. The hypothesis on C_φ implies that $\|{C_\varphi}^*\| \leq 1$. Thus, one has

$$h(|\varphi(0)|) = \|\delta_{\varphi(0)}\| = \|{C_\varphi}^*(\delta_0)\| \leq \|\delta_0\| = h(0).$$

As h is strictly increasing, this gives the desired condition. □

Theorem 3.4 *Assume that* h *is strictly increasing. Let* $\Phi : \mathcal{F}_C(\mathsf{X}) \longrightarrow \mathcal{F}_C(\mathsf{X})$ *be semi-multiplicative and non-constant such that* $\Phi(I)$ *is bijective. The following are equivalent.*

(1) $\|\Phi(T)\| = \|T\|$, *for all* $T \in \mathcal{F}_C(\mathsf{X})$;

(2) $\|\Phi(I)\| = 1$ *and* $\|\Phi(\delta_z)\| = \|\delta_z\|$, *for all* $z \in \mathsf{D}$;

(3) *there exists a unique couple of rotations* ρ *and* θ *of* D *such that*

$$\Phi(T) = C_\theta T C_\rho, \quad \text{for all } T \in \mathcal{F}_C(\mathsf{X}).$$

Proof. As in the proof of Theorem 3.1, we only need to show '(2) $\Longrightarrow$ (3)'. By hypothesis, there is $U \in \mathcal{F}_C(\mathsf{X})$ such that $U\Phi$ is multiplicative. The latter map cannot be constant, otherwise h would be so. On the other hand, since

$$\Phi(I) = \Phi(I^2) = \Phi(I)U\Phi(I),$$

from the bijectivity of $\Phi(I)$, it follows that U is bijective too and $U^{-1} = \Phi(I)$. According to [3, Corollary], U is necessarily induced by an automorphism σ of D and U^{-1} by σ^{-1}. Moreover,

$$\|C_{\sigma^{-1}}\| = \|U^{-1}\| = \|\Phi(I)\| = 1.$$

Hence, by Proposition 3.3, $\sigma^{-1}(0) = 0$. Consequently, σ is a rotation of D so that $U = C_\sigma$ is an isometry, according to the rotations property for the space X. This makes the map $U\Phi$ preserve the norm of any δ_z. Therefore, by Theorem 3.1, it is given by: $U\Phi(T) = C_{\rho^{-1}}TC_\rho$, where ρ is a rotation of D. Now, by setting $\theta = \rho^{-1} \circ \sigma^{-1}$, one clearly gets a rotation of D and from the second identity satisfied by the map $U\Phi$, one can deduce the desired expression of $\Phi(T)$. The uniqueness of (ρ, θ) can be shown the same way as for Theorem 3.1. □

From the previous theorem, one can easily derive the following.

Corollary 3.5 *Under the same hypothesis as in Theorem* 3.4*, if* $\|\Phi(I)\| = 1$ *and* $\|\Phi(\delta_z)\| = \|\delta_z\|$*, for all* $z \in \mathsf{D}$*, then* Φ *is bijective.*

Next, we recall that for a bounded operator T on a Banach space with $T \neq 0$, the *reduced minimum modulus* of T is defined by

$$\gamma(T) = \inf\Big\{\|T(x)\| : x \in \mathsf{X}, \operatorname{dist}\big(x, \mathsf{Ker}(T)\big) = 1\Big\}.$$

For more details about the reduced minimum modulus see [4, 7, 8]. In the special case where T is of rank one, it is not difficult to see that $\gamma(T) = \|T\|$. So, one can get analogous versions of Theorems 3.1 and 3.4, for the reduced minimum modulus, provided that X is a Banach space. Here is how they can be stated.

Proposition 3.6 *Under the same hypothesis as in Theorem* 3.1*, the following are equivalent.*

(1) $\gamma(\Phi(T) = \gamma(T)$, *for all* $T \in \mathcal{F}_C(\mathsf{X})$;

(2) $\gamma(\Phi(\delta_z)) = \gamma(\delta_z)$, *for all* $z \in \mathsf{D}$;

(3) *there exists a unique rotation* ρ *of* D *such that*

$$\Phi(T) = C_{\rho^{-1}} T C_\rho, \ \ \forall\, T \in \mathcal{F}_C(\mathsf{X}).$$

Proposition 3.7 *Under the same hypothesis as in Theorem* 3.4*, the following are equivalent.*

(1) $\gamma(\Phi(T)) = \gamma(T)$, $\forall T \in \mathcal{F}_C(\mathsf{X})$;

(2) $\gamma(\Phi(I)) = 1$ *and* $\gamma(\Phi(\delta_z)) = \gamma(\delta_z)$, $\forall z \in \mathsf{D}$;

(3) *there exist a unique couple of rotations* ρ *and* θ *of* D *such that*

$$\Phi(T) = C_\theta T C_\rho, \ \ \forall\, T \in \mathcal{F}_C(\mathsf{X}).$$

Proof. We only need to show '(2) $\Longrightarrow$ (3)'. Using the same argument as in the proof of Theorem 3.4 and taking into account Theorems 1.3 and 1.4, one obtains

$$U\Phi(\delta_z) = \delta_{\rho(z)} \quad \text{and} \quad C_\rho U \Phi(T) = T C_\rho,$$

for all $z \in \mathsf{D}$ and $T \in \mathcal{F}_C(\mathsf{X})$, with ρ and U given as in that proof. As $U \in \widetilde{\mathcal{F}_C}(\mathsf{X})$, it follows that $U\Phi(\delta_z) = \Phi(\delta_z) = \delta_{\rho(z)}$ for all $z \in \mathsf{D}$. This gives

$$\gamma(U\Phi(\delta_z)) = \gamma(\Phi(\delta_z)) = \gamma(\delta_z), \quad \text{for all } z \in \mathsf{D}.$$

Therefore, according to Proposition 3.6, applied with the multiplicative map $U\Phi$, ρ is forced to be a rotation of D and then

$$U\Phi(T) = C_{\rho^{-1}} T C_\rho, \ \text{ for all } \ T \in \mathcal{F}_C(\mathsf{X}).$$

On the other hand, one has

$$\|U\|^{-1} = \gamma(U^{-1}) = \gamma(\Phi(I)) = 1.$$

Thus, $\|U\| = 1$ and then, by Proposition 3.3, the symbol of U is a rotation. We end the proof the same way as that of Theorem 3.4. (Note that U is in fact an onto isometry. To see this, one can also show that $\gamma(U) = 1$). □

Remark.
We recall that in the case where h is strictly increasing, we have determined the preservers of $\mathcal{F}_e(\mathsf{X})$ contracting the norm by symbols fixing 0 (see [6]). For the set $\mathcal{F}_C(\mathsf{X})$, it is not difficult to see (by following the main ideas of the previous proofs) that the norm-contracting non-constant multiplicative (semi-multiplicative) preservers are necessarily among those given in Corollary 1.5 (Proposition 2.9) with φ vanishing at 0 (A and B induced by symbols vanishing at 0). However, the converse may not occur. To see this, one can consider the

symbols $\lambda\mathsf{p}_1$ ($\lambda \in \mathsf{D}\backslash\{0\}$) and the Hardy spaces H^p ($1 \le p < \infty$). From the non-trivial result in [1] giving the exact value of $\|C_{s\mathsf{p}_1+t}\|$ on H^p, one can observe the strict decrease of this norm as a function of $|t|^2$ to get

$$\|\Phi(C_{s\mathsf{p}_1+t})\| = \|C_{s\mathsf{p}_1+\lambda t}\| > \|C_{s\mathsf{p}_1+t}\|$$

for all $s, t \in \mathsf{D}\backslash\{0\}$, such that $|s| + |t| \le 1$, where Φ is the multiplicative preserver of $\mathcal{F}_C(H^p)$ related to the symbol $\varphi = \lambda\mathsf{p}_1$. Nevertheless, in the general case where no topology is required on X, one can verify that the nullity condition at 0, for the symbols, characterizes all the corresponding preservers leaving invariant the set of all composition operators with symbols fixing 0. Back to the space H^p, such a set is also the subclass of $\mathcal{F}_C(H^p)$ consisting of all contractions (consequence of the Littlewood subordination principle). All of this naturally leads to the following questions :

Question 1 : How can one strengthen the nullity condition at 0 to characterize the norm-contracting preservers of $\mathcal{F}_C(\mathsf{X})$?

Question 2 : Do contractive members of $\mathcal{F}_C(\mathsf{X})$ determine, as in the special case of H^p, all its multiplicative (semi-multiplicative) preservers leaving invariant its subset of all contractions ?

References

[1] C. C. Cowen, Linear fractional composition operators on H^2, In. Equ. Oper. Theory, 11 (1988) 151-160.

[2] C. C. Cowen, B. D. MacCluer , Composition Operators on Spaces of Analytic Functions, CRC Press, Boca Raton 1995.

[3] G. Hoever, Two classroom proof concerning composition operators, Integral Equations Operator Theory, 27 (1997) 493-496.

[4] S. Goldberg, Unbounded Linear operators, McGraw-Hill, New York, 1966.

[5] A. A. Jafarian, A. R. Sourour, Spectrum-preserving linear maps, J. Funct. Anal., 66, (1986) 255-261.

[6] N. Jaoua, H. Skhiri, Evaluation-functional-preserving maps, Expo. Math. 27 (2009) 211-226.

[7] T. Kato, Perturbation theory for nullity, deficiency, and other quantities of linear operators, J. Anal. Math. 6 (1958), 261–322.

[8] T. Kato, Perturbation theory for linear operators, Springer-Verlag, Berlin, 1966.

[9] J. H. Shapiro, Composition Operators and Classical Function Theory, Springer-Verlag, New York (1993).

[10] H. Skhiri, A Mbekhta's conjecture for reduced minimum modulus preserving, Acta. Sci. Math. (Szeged), 74 (2008), 853-862.

[11] H. Skhiri, Reduced minimum modulus preserving in Banach space, Integral Equations Operator Theory 62 (2008), 137-148.

[12] H. Skhiri, Linear maps preserving the minimum and surjectivity moduli of Hilbert space operators, J. Math. Anal. Appl. 358 (2009), 320-326.

[13] H. Skhiri, Minimum and surjectivity moduli preserving in Banach space, Acta Appl. Math. 112, (2010) 347-356.

[14] A. R. Sourour, Invertibility preserving linear maps on $\mathcal{L}(\mathsf{X})$, Trans. Amer. Soc., 348 , (1996) 13-30.

Received: June, 2011