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Two Degree non Homogeneous Differential Equations with Polynomial Coefficients

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Abstract

We solve some forms of non homogeneous differential equations using a new function u_g which is integral-closed form solution of a non homogeneous second order ODE with linear coefficients. The non homogeneous part is an arbitrary function of $L_2(\mathbf{R})$. Using this function u_g as a base we give in closed integral form solutions of several two degree DE.

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1 Introduction

We will solve the equation

$$(a_1x+b_1)f''(x)+(a_2x+b_2)f'(x)+(a_3x+b_3)f(x)=g(x) \tag{1}$$

where f, $g \in L_2(\mathbf{R})$ and a_1, a_2, a_3, b_1, b_2, b_3 are constants in $\mathbf{R}$.
We call the solution u_g, and using this solution we try to solve other general differential equations.
The solution of (1) (function u_g), follow using the Fourier transform and its properties. The result is a quite complicated integral. However the Fourier transform of this integral is not so complicated and satisfies as we will see a simple first order differential equation. We also consider and solve completely a class of differential equations having two degree polynomial coefficients.
We also give examples and applications. Special cases of u_g are known functions, such as Hermite polynomials, Bessel functions, confluent hypergeometric functions and other.

2 Preliminary Notes

Let the Fourier Transform of a function f of $L_2(\mathbf{R})$ is

$$\widehat{f}(\gamma) = \int_{-\infty}^{\infty} f(t)e^{-it\gamma}dx$$

the Inverse Fourier Transform is

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\gamma)e^{i\gamma x}d\gamma$$

Then it is known (integration by parts)

$$\int_{-\infty}^{\infty} f(x)x^n e^{-ix\gamma}dx = i^n(\widehat{f})^{(n)}(\gamma). \tag{2}$$

$$(\widehat{f^{(n)}})(\gamma) = (i\gamma)^n\widehat{f}(\gamma). \tag{3}$$

$$\int_{-\infty}^{\infty} f'(x)A(x)e^{-ix\gamma}dx =$$

$$= \int_{-\infty}^{\infty} f(x)A'(x)e^{-ix\gamma}dx + (-i\gamma)\int_{-\infty}^{\infty} f(x)A(x)e^{-ix\gamma}dx. \tag{4}$$

$$\int_{-\infty}^{\infty} f''(x)A(x)e^{-ix\gamma} = \int_{-\infty}^{\infty} f(x)A''(x)e^{-ix\gamma}dx+$$

$$+2(-i\gamma)\int_{-\infty}^{\infty} f(x)A'(x)e^{-ix\gamma}dx + (-i\gamma)^2\int_{-\infty}^{\infty} f(x)A(x)e^{-ix\gamma}dx. \tag{5}$$

(see [2]).

3 Main Results

These are the main results of the paper.

Theorem 3.1 *When f, $g \in L_2(\mathbf{R})$ and $\lim_{|x|\to\infty}|f(x)x^{2+\epsilon}| = 0$, $\epsilon > 0$, equation (1) can reduced to*

$$(-ia_1\gamma^2 + a_2\gamma + ia_3)\frac{\widehat{f}(\gamma)}{d\gamma} + (-b_1\gamma^2 - 2ia_1\gamma + ib_2\gamma + a_2 + b_3)\widehat{f}(\gamma) = \widehat{g}(\gamma) \tag{6}$$

which is solvable.

Proof.
Take the Fourier Transform in both sides of (1) and use relations (2),(3),(4),(5). We immediately arrive to (6). But (6) is completely solvable DE of first degree.

After solving it we use the inverse Fourier transform to get the solution, which we call it

$$f(x) = u_g\left[\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}; x\right]. \tag{7}$$

and satisfies (6).

Note. When $g(x) = 0$, the solution is very simple as someone can see is

$$\widehat{u_0}\left[\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}; \gamma\right] =$$

$$= \exp\left(-\int \frac{-b_1\gamma^2 - 2ia_1\gamma + ib_2\gamma + a_2 + b_3}{-ia_1\gamma^2 + a_2\gamma + ia_3} d\gamma\right) \tag{8}$$

Theorem 3.2 *The DE*

$$(ax^2 + bx + c)y''(x) + (kx^2 + lx + m)y'(x) + (tx^2 + rx + s)y(x) = g(x) \tag{9}$$

is solvable if

$$k = \frac{a\left(4a\Delta - bl - \Delta l + 2am\right)}{-b^2 + 2ac - b\Delta} \tag{10}$$

$$s = 6a - \frac{2b^2}{c} + \frac{2b\Delta}{c} - l + \frac{bm}{c} - \frac{\Delta m}{c} + \frac{br}{2a} + \frac{\Delta r}{2a} - \frac{b^2 t}{2a^2} + \frac{ct}{a} - \frac{b\Delta t}{2a^2} \tag{11}$$

and $\Delta = \sqrt{b^2 - 4ac}$
The solution is

$$y(x) = \frac{a_1 x + b_1}{ax^2 + bx + c} u_g[\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}; x] \tag{12}$$

where

$$a_1 = \frac{ab_3\left(b + \Delta\right)}{\left(b + \Delta\right)k + 2a(2a - l + s)} \tag{13}$$

$$a_2 = \frac{b_3\left(b + \Delta\right)k}{\left(b + \Delta\right)k + 2a(2a - l + s)} \tag{14}$$

$$a_3 = \frac{b_3\left(b + \Delta\right)t}{\left(b + \Delta\right)k + 2a(2a - l + s)} \tag{15}$$

$$b_1 = \frac{2ab_3c}{\left(b + \Delta\right)k + 2a(2a - l + s)} \tag{16}$$

$$b_2 = -\frac{b_3\left(b + \Delta\right)\left(4a^2 + bk + \Delta k - 2al\right)}{2a\left(4a^2 + bk + \Delta k - 2al + 2as\right)} \tag{17}$$

Proof. We proceed as in Proposition 1.
Set

$$y(x)=\frac{a_1x+b_1}{ax^2+bx+c}Y(x) \tag{18}$$

We arrive in a equation

$$(a_1x+b_1)Y''(x)+(a_2x+b_2)Y'(x)+(a_3x+b_3)Y(x)=g(x), \tag{19}$$

in which the relation of the coefficients of the two equations (9) and (19) is:

$$b=\frac{ab_1^2+a_1^2c}{a_1b_1}\ ,\ k=\frac{aa_2}{a_1}\ ,\ l=2a+\frac{ab_2}{a_1}+\frac{a_2c}{b_1},\ t=\frac{aa_3}{a_1}$$

$$m=\frac{2ab_1^2+a_1b_2c}{a_1b_1}\ ,\ r=\frac{aa_2b_1+ab_1b_3+a_1a_3c}{a_1b_1}\ ,\ s=\frac{ab_1b_2+a_1b_3c}{a_1b_1}$$

Solving the above system we get the proof.
i) Theorem 2 prove that, a two degree ODE with second degree polynomial coefficients is 6-th parameter solvable DE and admits a general solution $y(x)=u_g(x)$ in a closed integral form. Thus if we give any values we want in (9) then it is solvable always except for two values of the coefficients (here) k and s, which are determined by the others.
ii) A way to find some special cases of solutions of the DE (9) is with the command 'SolveAlways' of the program Mathematica.

Example 1. The equation

$$\left(x^2+3x+1/3\right)y''(x)+\left(x^2\frac{207-25\sqrt{69}}{2}+2x+3\right)y'(x)+$$

$$+y(x)\left(\frac{-x^2}{2}+x+\frac{-233+41\sqrt{69}}{12}\right)=0 \tag{20}$$

is equivalent to the second order, first degree polynomial coefficients ODE

$$-(95-23\sqrt{69}+(9+\sqrt{69})x)Y(x)+2(2\sqrt{69}+(69-9\sqrt{69})x)Y'(x)+$$

$$+(2+(9+\sqrt{69})x)Y''(x)=0, \tag{21}$$

which is solvable according to Theorem 1.

$$y(x)=\frac{9(2+(9+\sqrt{69})x)}{(-95+23\sqrt{69})(1+9x+3x^2)}\times$$

$$\times u_0\left[\left\{6+\sqrt{69},2(69-9\sqrt{69}),-9-\sqrt{69}\right\},\left\{2,4\sqrt{69},-95+23\sqrt{69}\right\}\right]$$

where u_0 is given by (8).

Theorem 3.3 *The DE*

$$\left[ax^2 + \frac{ab_1}{a_1}x\right] y''(x) + \left[kx^2 + (2a+s)x + \frac{2ab_1}{a_1}\right] y'(x) + \left[tx^2 + rx + s\right] y(x) = g(x), \tag{22}$$

with $g \in L_2(\mathbf{R})$, *have a closed integral form solution*

$$y(x) = u_g\left[\left\{a_1, \frac{a_1 k}{a}, \frac{a_1 t}{a}\right\}, \left\{b_1, \frac{a_1 s}{a}, \frac{-a_1 k + a_1 r}{a}\right\}; x\right] \tag{23}$$

Applications in the case of linear coefficients

1.

The homogeneous of the differential equation

$$2xy''(x) + y'(x) - 2y(x) = g(x) \tag{24}$$

is

$$2xy''(x) + y'(x) - 2y(x) = 0 \tag{25}$$

and have solutions

$$y_1(x) = \cosh(2\sqrt{x})\ ,\ y_2(x) = \sinh(2\sqrt{x})$$

With our method (24) have solution

$$y(x) = u_g[\{2, 0, 0\}, \{1, 1, -2\}; x]$$

where

$$\widehat{y}(w) = \frac{e^{-i/w}}{w^{3/2}}\left(C_1 + \int_c^w \frac{ie^{i/x}G(x)}{2\sqrt{x}}dx\right)$$

and $G(x) = \widehat{g}(x)$.

2.

The homogeneous of the differential equation

$$y''(x) - 2xy'(x) + 2ny(x) = g(x) \tag{26}$$

is

$$y''(x) - 2xy'(x) + 2ny(x) = 0 \tag{27}$$

and have solution

$$y_1(x) = H_n(x),\ y_2(x) = {}_1F_1\left[-\frac{n}{2}, \frac{1}{2}; x^2\right]$$

With our method (26) have solution

$$y(x) = u_g[\{0, -2, 0\}, \{1, 0, 2n\}; x]$$

where

$$\widehat{y}(w) = e^{i(1+n)\arctan\left(\frac{2}{w}\right)} w^{-1+n} \left(4 + w^2\right)^{-\frac{1}{2}-\frac{n}{2}} [C_1 +$$

$$+ \int_c^w ie^{-i(1+n)\arctan\left(\frac{2}{x}\right)} G(x) x^{-n} (2i + x) \left(4 + x^2\right)^{\frac{1}{2}(-1+n)} dx]$$

and $G(x) = \widehat{g}(x)$.

3.

The homogeneous of the differential equation

$$xy''(x) + y(x) = g(x) \tag{28}$$

is

$$xy''(x) + y(x) = 0 \tag{29}$$

and have solution

$$y(x) = \sqrt{x}\left[C_1 J_1(2\sqrt{x}) + C_2 Y_1(2\sqrt{x})\right]$$

With our method (28) have solution

$$y(x) = u_g[\{1, 0, 0\}, \{0, 0, 1\}; x]$$

where

$$\widehat{y}(w) = \frac{1}{w^2} e^{\frac{i}{w}} \left(C_1 + \int_c^w ie^{-\frac{i}{x}} G(x)\, dx\right)$$

and $G(x) = \widehat{g}(x)$.

Note.

In the above examples if we extract the general solutions and then set in the equations $G(x) = 0$, we arrive to the reduced homogeneous solution.

Applications for the two degree polynomials coefficients

1.

The homogeneous of the differential equation

$$x^2 y''(x) + [kx^2 + x(2+s)]y'(x) + (tx^2 + rx + s)y(x) = g(x) \tag{30}$$

is

$$x^2 y''(x) + [kx^2 + x(2+s)]y'(x) + (tx^2 + rx + s)y(x) = 0 \tag{31}$$

and have solution
$y(x) =$

$$= \frac{C_1}{x} e^{-kx/2-\sqrt{k^2-4t}x/2} U\left(\frac{-2r+k(2+s)+s\sqrt{k^2-4t}}{2\sqrt{k^2-4t}}, s, x\sqrt{k^2-4t}\right) +$$

$$+\frac{C_2}{x} e^{-kx/2-\sqrt{k^2-4t}x/2} L\left(\frac{2r-k(2+s)-s\sqrt{k^2-4t}}{2\sqrt{k^2-4t}}, -1+s, x\sqrt{k^2-4t}\right)$$

Where $L_n^a(x) = L(n, a, x)$ is the generalized Laguerre polynomial and $U(a, b, x)$ is confluent hypergeometric function.
With our method (30) have solution

$$y(x) = \frac{a_1}{x} u_g[\{a_1, a_1 k, a_1 t\}, \{0, a_1 s, -a_1 k + a_1 r\}; x]$$

2.
The equation

$$x(1-x)y''(x)+(kx^2+lx-2-k-l)y'(x)+[tx^2+(-s+2+2k+l-t)x+s]y(x) = g(x)$$

have solution

$$y(x) = \frac{a_1}{1-x}\times$$

$$\times u_g[\{a_1, -a_1 k, -a_1 t\}, \{0, -2a_1 - a_1 k - a_1 l, -2a_1 - a_1 k - a_1 l + a_1 s\}; x]$$

Example of the above equation we get if we choose $b_1 = 0$, $a_2 = 0$, $b_2 = -2a_1 - a_1 l$, $a_3 = -a_1 t$, $b_3 = -2a_1 - a_1 l + a_1 s$ and $k = 0$, then

$$\widehat{y}(w) = e^{-\frac{i(2+l-s)}{w}} w^{-4-l} \left(C_1 + \int_c^w \frac{ie^{\frac{i(2+l-s)}{x}} G(x) x^{2+l}}{a_1} dx\right)$$

or

$$y(x) = \frac{a_1}{1-x} u_g[\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}; x]$$

is the solution of

$$(1-x)xy''(x) + (lx-2-l)y'(x) + [tx^2 + (2+l-t)x + s]y(x) = g(x)$$

3.
The equation

$$x(1-x)y''(x) + (kx^2 + lx + m)y'(x) + (tx^2 + rx + s)y(x) = g(x) \qquad (32)$$

is solvable with the u_g functions when $k + l + m + 2 = 0$.

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