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Gülicher' theorem in the Poincaré disc model of Hyperbolic Geometry

Cătălin Barbu

"Vasile Alecsandri" National College
Vasile Alecsandri 37,
600011, Bacău, Romania
kafka_mate@yahoo.com

Laurian-Ioan Pişcoran

North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122, Baia Mare, Romania
plaurian@yahoo.com

Abstract

In this note, we present the hyperbolic Gülicher theorem in the Poincaré disc model of hyperbolic geometry.

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1 Introduction

Hyperbolic geometry appeared in the first half of the 19^{th} century as an attempt to understand Euclid's axiomatic basis of geometry. It is also known as a type of non-euclidean geometry, being in many respects similar to euclidean geometry. Hyperbolic geometry includes similar concepts as distance and angle. Both these geometries have many results in common but many are different. Several useful models of hyperbolic geometry are studied in the literature as, for instance, the Poincaré disk and ball models, the Poincaré half-plane model, and the Beltrami-Klein disk and ball models, etc. Following [4] and [5] and earlier discoveries, the Beltrami-Klein model is also known as the Einstein relativistic velocity model. Here, in this study, we present a Proof of Gülicher's theorem in the Poincaré disk model of hyperbolic geometry. Gülicher's theorem states that if $Q_1Q_2Q_3$ is the cevian triangle of point Q with respect to the triangle $P_1P_2P_3$, and $R_1R_2R_3$ is the cevian triangle of point R with respect to the triangle $Q_1Q_2Q_3$, then the lines P_1R_1, P_2R_2, and P_3R_3 are concurrent [3].

Let D denote the complex unit disk in complex z - plane, i.e.

$$D = \{z \in \mathbb{C} : |z| < 1\}.$$

The most general Möbius transformation of D is

$$z \to e^{i\theta} \frac{z_0 + z}{1 + \overline{z_0} z} = e^{i\theta}(z_0 \oplus z),$$

which induces the Möbius addition $\oplus$ in D, allowing the Möbius transformation of the disk to be viewed as a Möbius left gyrotranslation

$$z \to z_0 \oplus z = \frac{z_0 + z}{1 + \overline{z_0} z}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $z, z_0 \in D$, and $\overline{z_0}$ is the complex conjugate of z_0. Let $Aut(D, \oplus)$ be the automorphism group of the grupoid $(D, \oplus)$. If we define

$$gyr : D \times D \to Aut(D, \oplus), gyr[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\overline{b}}{1 + \overline{a}b},$$

then is true gyrocommutative law

$$a \oplus b = gyr[a, b](b \oplus a).$$

A gyrovector space $(G, \oplus, \otimes)$ is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) $gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\, gyr[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a} \cdot \mathbf{b}$ for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(G1) $1 \otimes \mathbf{a} = \mathbf{a}$

(G2) $(r_1 + r_2) \otimes \mathbf{a} = r_1 \otimes \mathbf{a} \oplus r_2 \otimes \mathbf{a}$

(G3) $(r_1 r_2) \otimes \mathbf{a} = r_1 \otimes (r_2 \otimes \mathbf{a})$

(G4) $\frac{|r| \otimes \mathbf{a}}{\|r \otimes \mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$

(G5) $gyr[\mathbf{u}, \mathbf{v}](r \otimes \mathbf{a}) = r \otimes gyr[\mathbf{u}, \mathbf{v}]\mathbf{a}$

(G6) $gyr[r_1 \otimes \mathbf{v}, r_1 \otimes \mathbf{v}] = 1$

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one dimensional "vectors"

$$\|G\| = \{\pm \|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R},$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(G7) $\|r \otimes \mathbf{a}\| = |r| \otimes \|\mathbf{a}\|$

(G8) $\|\mathbf{a} \oplus \mathbf{b}\| \leq \|\mathbf{a}\| \oplus \|\mathbf{b}\|$

Definition 1.1 *The hyperbolic distance function in D is defined by the equation*

$$d(a, b) = |a \ominus b| = \left| \frac{a - b}{1 - \overline{a}b} \right|.$$

Here, $a \ominus b = a \oplus (-b)$, for $a, b \in D$.

Theorem 1.2 (***The law of gyrosines in Möbius gyrovector spaces***). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and with gyroangles α, β, and γ at the vertices A, B, and C. Then $\frac{a_\gamma}{\sin\alpha} = \frac{b_\gamma}{\sin\beta} = \frac{c_\gamma}{\sin\gamma}$, where $v_\gamma = \frac{v}{1 - \frac{v^2}{s^2}}$.*

(see [4], p. 294)

For further details we refer to the recent book of A.Ungar [4].

Theorem 1.3 (***Transversal Theorem for Gyrotriangles***). *Let D be on gyroside BC, and l is a gyroline not through any vertex of a gyrotriangle ABC such that l meets AB in M, AC in N, and AD in P, then*

$$\frac{(BD)_\gamma}{(CD)_\gamma} \cdot \frac{(CA)_\gamma}{(NA)_\gamma} \cdot \frac{(NP)_\gamma}{(MP)_\gamma} \cdot \frac{(MA)_\gamma}{(BA)_\gamma} = 1.$$

(see [2])

Theorem 1.4 (***The Ceva's Theorem for Hyperbolic Gyrotriangle***). *If M is a point not on any side of an gyrotriangle $A_1A_2A_3$ such that A_3M and A_1A_2 meet in P, A_2M and A_3A_1 in Q, and A_1M and A_2A_3 meet in R, then*

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R_1)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1$$

(see [1])

Theorem 1.5 (***Converse of Ceva's Theorem for Hyperbolic Gyrotriangle***). *If P lies on the gyroline A_1A_2, R on A_2A_3, and Q on A_3A_1 such that*

$$\frac{(A_1P)_\gamma}{(A_2P)_\gamma} \cdot \frac{(A_2R)_\gamma}{(A_3R_1)_\gamma} \cdot \frac{(A_3Q)_\gamma}{(A_1Q)_\gamma} = 1,$$

and two of the gyrolines A_1R, A_2Q and A_3P meet, then all three are concurrent.

(see [1])

2 Main Results

In this section, we prove the Gülicher's theorem in the Poincaré disk model of hyperbolic geometry.

Theorem 2.1 (***The Gülicher's Theorem for Hyperbolic Gyrotriangle***). *Let $Q_1Q_2Q_3$ be the cevian gyrotriangle of gyropoint Q with respect to the gyrotriangle $P_1P_2P_3$, and Q is located inside the gyrotriangle $P_1P_2P_3$. Let $R_1R_2R_3$ be the cevian gyrotriangle of gyropoint R with respect to the gyrotriangle $Q_1Q_2Q_3$, and R is located inside the gyrotriangle $Q_1Q_2Q_3$. Then the gyrolines P_1R_1, P_2R_2, and P_3R_3 are concurrent.*

Proof. Let X, Y, Z be the intersection points of the gyrolines P_1R_1, P_2R_2, and P_3R_3 with gyroline P_2P_3, P_3P_1, and P_1P_2, respectively (See Figure 1).

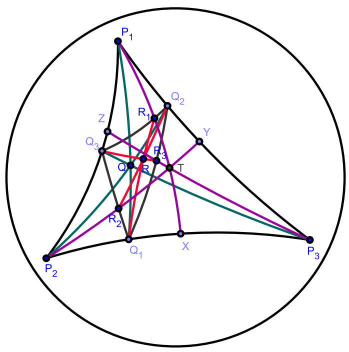

Figure 1

If we use a Theorem 1.3 in the gyrotriangle $P_1P_2P_3$ for the gyrolines P_1R_1X, P_2R_2Y, and P_3R_3Z, we have

$$\frac{(P_2X)_\gamma}{(P_3X)_\gamma} = \frac{(P_1Q_2)_\gamma}{(P_1Q_3)_\gamma} \cdot \frac{(R_1Q_3)_\gamma}{(R_1Q_2)_\gamma} \cdot \frac{(P_1P_2)_\gamma}{(P_1P_3)_\gamma} \tag{1}$$

and

$$\frac{(P_3Y)_\gamma}{(P_1Y)_\gamma} = \frac{(P_2Q_3)_\gamma}{(P_2Q_1)_\gamma} \cdot \frac{(R_2Q_1)_\gamma}{(R_2Q_3)_\gamma} \cdot \frac{(P_2P_3)_\gamma}{(P_2P_1)_\gamma}, \tag{2}$$

and

$$\frac{(P_1Z)_\gamma}{(P_2Z)_\gamma} = \frac{(P_3Q_1)_\gamma}{(P_3Q_2)_\gamma} \cdot \frac{(R_3Q_2)_\gamma}{(R_3Q_1)_\gamma} \cdot \frac{(P_3P_1)_\gamma}{(P_3P_2)_\gamma}. \tag{3}$$

Multiplying relations (1), (2), and (3) member by member, we obtain

$$\frac{(P_2X)_\gamma}{(P_3X)_\gamma} \cdot \frac{(P_3Y)_\gamma}{(P_1Y)_\gamma} \cdot \frac{(P_1Z)_\gamma}{(P_2Z)_\gamma}$$

$$= \left[\frac{(P_1Q_2)_\gamma}{(P_1Q_3)_\gamma} \cdot \frac{(P_2Q_3)_\gamma}{(P_2Q_1)_\gamma} \cdot \frac{(P_3Q_1)_\gamma}{(P_3Q_2)_\gamma}\right] \cdot \left[\frac{(R_1Q_3)_\gamma}{(R_1Q_2)_\gamma} \cdot \frac{(R_2Q_1)_\gamma}{(R_2Q_3)_\gamma} \cdot \frac{(R_3Q_2)_\gamma}{(R_3Q_1)_\gamma}\right]. \tag{4}$$

If we use a Theorem 1.4 for the gyrotriangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ with concurrent cevians P_1Q_1, P_2Q_2, P_3Q_3 and Q_1R_1, Q_2R_2, Q_3R_3 respectively, we get

$$\frac{(P_1Q_2)_\gamma}{(P_3Q_2)_\gamma} \cdot \frac{(P_2Q_3)_\gamma}{(P_1Q_3)_\gamma} \cdot \frac{(P_3Q_1)_\gamma}{(P_2Q_1)_\gamma} = 1 \tag{5}$$

and

$$\frac{(R_1Q_3)_\gamma}{(R_1Q_2)_\gamma} \cdot \frac{(R_2Q_1)_\gamma}{(R_2Q_3)_\gamma} \cdot \frac{(R_3Q_2)_\gamma}{(R_3Q_1)_\gamma} = 1. \tag{6}$$

From (4), (5), and (6) we obtain

$$\frac{(P_2X)_\gamma}{(P_3X)_\gamma} \cdot \frac{(P_3Y)_\gamma}{(P_1Y)_\gamma} \cdot \frac{(P_1Z)_\gamma}{(P_2Z)_\gamma} = 1, \tag{7}$$

since, from theorem 1.5, that the gyrolines P_1R_1, P_2R_2, and P_3R_3 are concurrent.

Lemma 2.2 (***The Gyrotriangle Bisector Theorem***). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(V_s, \oplus, \otimes)$ with vertices $A, B, C \in V_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbf{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$, $\mathbf{a} = \ominus B \oplus C$, $\mathbf{b} = \ominus C \oplus A$, $\mathbf{c} = \ominus A \oplus B$, $a = \|\mathbf{a}\|$, $b = \|\mathbf{b}\|$, $c = \|\mathbf{c}\|$, and let D be a point lying on side BC of the gyrotriangle such that AD is a bisector of gyroangle $\angle BAC$. Then*

$$\frac{(DB)_\gamma}{(DC)_\gamma} = \frac{(AB)_\gamma}{(AC)_\gamma},$$

where $v_\gamma = \frac{v}{1-\frac{v^2}{s^2}}$.

Proof. Denote by $\alpha_1 = \angle BAD$, and $\alpha_2 = \angle CAD$. Because AD is a bisector of gyroangle $\angle BAC$, we get that $\sin\alpha_1 = \sin\alpha_2$ (see Figure 2).

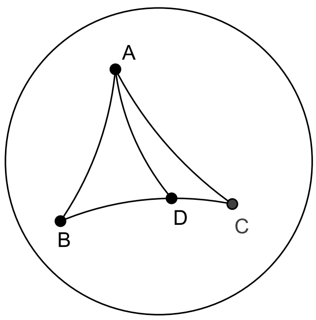

Figure 2

If we use a Theorem 1.2 in the gyrotriangles ABC, ABD, and ACD we have

$$\frac{\sin C}{\sin B} = \frac{(AB)_\gamma}{(AC)_\gamma}, \tag{8}$$

and

$$\frac{\sin \alpha_1}{\sin B} = \frac{(DB)_\gamma}{(DA)_\gamma}, \tag{9}$$

and

$$\frac{\sin \alpha_2}{\sin C} = \frac{(DC)_\gamma}{(DA)_\gamma}. \tag{10}$$

If ratios the equations (9) and (10) among themselves, respectively, then

$$\frac{\sin C}{\sin B} = \frac{(DB)_\gamma}{(DC)_\gamma}. \tag{11}$$

From the relations (8) and (11) the conclusion follows.

Theorem 2.3 *Let $Q_1Q_2Q_3$ be the cevian gyrotriangle of gyropoint Q with respect to the gyrotriangle $P_1P_2P_3$, and Q is located inside the gyrotriangle $P_1P_2P_3$. If the bisectors of gyroangles of gyrotriangle $P_1P_2P_3$ meet the gyrosides Q_2Q_3, Q_3Q_1, and Q_1Q_3 at the gyropoints R_1, R_2, and R_3, respectively, then the gyrolines Q_1R_1, Q_2R_2, and Q_3R_3 are concurrent.*

Proof. If we use a Theorem 1.4 in the gyrotriangle $P_1P_2P_3$ for concurrent cevians P_1Q_1, P_2Q_2, P_3Q_3 (see Figure 1), we get

$$\frac{(P_1Q_2)_\gamma}{(P_3Q_2)_\gamma} \cdot \frac{(P_2Q_3)_\gamma}{(P_1Q_3)_\gamma} \cdot \frac{(P_3Q_1)_\gamma}{(P_2Q_1)_\gamma} = 1. \tag{12}$$

Now, we use Lemma 2.2 in the gyrotriangles $P_1Q_2Q_3, P_2Q_3Q_1$, and $P_3Q_1Q_2$ we have

$$\frac{(P_1Q_2)_\gamma}{(P_1Q_3)_\gamma} = \frac{(R_1Q_2)_\gamma}{(R_1Q_3)_\gamma}, \tag{13}$$

and

$$\frac{(P_2Q_3)_\gamma}{(P_2Q_1)_\gamma} = \frac{(R_2Q_3)_\gamma}{(R_2Q_1)_\gamma}, \tag{14}$$

and

$$\frac{(P_3Q_1)_\gamma}{(P_3Q_2)_\gamma} = \frac{(R_3Q_1)_\gamma}{(R_3Q_2)_\gamma}. \tag{15}$$

Multiplying the relations (13), (14), and (15), and we use the relation (12) we obtain

$$\frac{(R_1Q_2)_\gamma}{(R_1Q_3)_\gamma} \cdot \frac{(R_2Q_3)_\gamma}{(R_2Q_1)_\gamma} \cdot \frac{(R_3Q_1)_\gamma}{(R_3Q_2)_\gamma} = 1, \tag{16}$$

and by Theorem 1.5, we get that the gyrolines Q_1R_1, Q_2R_2, and Q_3R_3 are concurrent.

Many of the theorems of Euclidean geometry are relatively similar form in the Poincaré model of hyperbolic geometry, Gülicher's theorem is an example in this respect. In the Euclidean limit of large s, $s \to \infty$, v_γ reduces to v, so Gülicher's theorem for hyperbolic triangle reduces to the Gülicher's theorem of euclidian geometry.

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