

## SET THEORY IN LINEAR ALGEBRA

HAMIDE DOGAN  
University of Texas at El Paso,  
Department of Mathematical Sciences  
El Paso, TX 79968  
hdogan@utep.edu

### Abstract

*Set theory language is an essential prerequisite for the cognition of linear algebra concepts. Many difficulties of cognition in linear algebra may be explained by the lack of mastery of set theory concepts. In the paper, an in depth discussion of documented categories of difficulties originated from set theory is provided.*

**Keywords:** Linear algebra, Set theory, Cognition.

**Mathematics Subject Classification:** 97C30, 97H60, 97E60

### 1. Introduction

We all witness cognitive difficulties with abstract mathematics concepts. Research indicates that formalism and the lack of set theory knowledge are among the reasons for learners' struggle with and mistakes in linear algebra concepts [2-7]. Dorier and Sierpinska [7] add the nature and historical background of linear algebra among the reasons. Formalism entails a wide range, from the use of notations and symbols to the structures used to represent ideas. One such representation tool is the language of set theory. Linear algebra makes use of set theory language quite often. For instance, vector

space concepts such as subspaces and spanning sets are often introduced through set theory-based representations. It is no doubt that one needs to have the knowledge of set theory as prerequisite for successful learning and understanding of linear algebra concepts. Dorier, Robert, Robinet and Rogalski in their paper [5] give various examples of a group of their students' work as testimony to the necessity and the importance of set theory knowledge in responding accurately to linear algebra questions. Referring to their students' incorrect responses, "...incorrect responses show both the lack of appropriation of the notions in question and the more or less inadequate mastery of set theory language" (p 90). Set theory is not only the foundation for linear algebra but also for abstract algebra. For instance, Brenton and Edwards [1] looked at their algebra students' work and come to the conclusion that the failure of good students with the concept of factor groups is due to the lack of understanding of the elements of quotient groups.

## **2. Set Theory in Cognition of Linear Algebra Concepts**

Clearly set theory language is an essential prerequisite for the cognition of linear algebra concepts. Many difficulties of cognition in linear algebra furthermore can be explained by the lack of the presence of the mastery of set theory concepts. Our investigations with linear algebra students (see for the details of our work at Dogan-Dunlap [3]) revealed that the lack of mastery of set theory knowledge may explain many of their misconceptions about linear algebra concepts. As a result of our work [3], we documented three main categories as likely aspects of set theory whose lack of seems to cause difficulties with the cognition of linear algebra concepts [3]:



real numbers [3]. This student may be aware of the real number condition yet he/she shows the lack of understanding of the particular condition being necessary but not sufficient to determine membership for the subset  $S$ .

Another group of students is observed to use a similar reasoning to identify the set  $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{R} \wedge x_1 \geq 0 \right\}$  to be a subspace of  $\mathbb{R}^2$ . Specifically, the sums of vector entries being real numbers were substantial for them to verify closure properties of  $W$  whereas ignoring the inequality-based characteristics of its members. This line of reasoning resulted in the incorrect identification of  $W$  being a subspace even though  $W$  does not hold the “*closure under scalar multiplication*” condition.

Other types of mistakes we observed that appear to be due to one’s inability to recognize the necessary and sufficient conditions determining the members of a set is the case where learners completely ignore any type of membership criteria or consider inaccurate descriptions. For instance, some responses displayed the lack of awareness of the existence of conditions determining the elements of the set  $W$  defined above. Typical responses had statements similar to “*because  $x_1$  and  $x_2$  exist in  $W$ , it is safe to say that  $x_1+x_2$  also exists in  $W$ .*” Clearly these responses indicate one’s deficiency in their knowledge of set theory.

One other type is revealed in the responses similar to “*Since both [meaning  $x_1$  and  $x_2$ ] are real numbers they could be a scalar multiple of each other with the matrix as shown  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  shows that  $x_2$  is a multiple of  $x_1$  and that it will span a line, not  $\mathbb{R}^2$  space.*”

Here it appears that the set  $W$  (defined above) is considered as  $\text{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$  thus its



becoming the elements of the solution set for the equation  $v^T x = 0$  where  $v = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -1 \end{bmatrix}$ . The

second response type points to inability to distinguish the set notation that includes all elements (infinitely many) from the notation that contains only finitely many vectors. Second response furthermore is the case of one's struggle to consider all the vectors satisfying the equation  $v^T x = 0$  as the elements of its solution set. The student who provided the particular response came to his/her set description initially observing and generalizing a single solution  $x_1=1, x_2=1, x_3=1$  and  $x_4=0$  of the equation to a solution set of vectors whose components  $x_1, x_2,$  and  $x_3$  having the same real number values and  $x_4$  taking any real number value (See the details of the student work in the next section under response 1).

Moreover considering only finitely many members of a set with infinitely many elements shows up in work with tasks similar to identifying the set of all values of a symbol that stands for the entries of a matrix, in turn will be invertible, nonsingular, or the matrix will represent a consistent system of linear equations. For example, some responses for the task "*find the values of 'a' such that the matrix  $A = \begin{bmatrix} a & 1 \\ a & 2 \end{bmatrix}$  is non-singular*" try out just a few values such as  $a=0, a=1$  and  $a=2$  and (after observing identity or non-identity form on the row reduced echelon form of  $A$  for each chosen values of "a"), come to conclusions similar to "*...for  $a=1$  and  $a=2$ , the matrix  $A$  is non-singular*" without regard to all the other non-zero real values of "a" that also result in a non-singular matrix. This kind of reasoning implies the lack of understanding of sets with infinitely many elements.



$\text{Ker}(T) = \{ t(-2, 1) \mid t \in \mathbb{R} \}$ . Therefore, a basis for  $\text{Ker}(T) = \{(-2, 1)\}$ .” This type of responses indicates a reasonable knowledge and an understanding of parametric representation but reveals the lack of awareness of connections between the parametric representations and the set notations of elements of sets. One can see that the particular response has a correct form for the parametric representation of the vectors of the kernel of the linear transformation  $T$  defined by the matrix  $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$  but transfers this form to a set notational form for the set,  $\text{Ker}(T)$ , inaccurately. Even though one (applying this kind of reasoning) may arrive at an accurate parametric representation, he/she may not be able to transfer his/her first representation to the other set theoretic algebraic structures.

We can furthermore observe struggle with set theory notation in tasks where the bases of sets are to be provided. We witnessed our students using set notational structures or the parametric representations of members of sets as bases. For example, when asked

to find a basis for the solution set of  $v^T x = 0$  with  $v = \begin{bmatrix} 1 \\ 2 \\ -3 \\ -1 \end{bmatrix}$ , some included responses

similar to the following excerpts:

**Response 1:**  $[1, 2, -3, -1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0 \dots$

$1 \cdot x_1 +$	$2 \cdot x_2 -$	$3 \cdot x_3 - 1 \cdot x_4 =$	$0$
$x_1 +$	$2x_2 -$	$3x_3 - x_4 =$	$0$
$1 +$	$2 -$	$3 - 0 =$	$0$
$x_1 = 1$	$x_2 = 1$	$x_3 = 1$	$x_4 = 0$

$\dots$  Thus a

basis would be  $\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_1 = x_2 = x_3, x_4 \in \mathbb{R} \right\}$ .





It is obvious that all three responses are displaying trouble working with the varying aspects of set theory concepts leading to inaccurate responses to the particular linear algebra task.

### 3. Conclusion

Many difficulties of cognition in linear algebra may be explained by the lack of mastery of set theory concepts. In this paper, we attempted to share student work from our investigations [3,4] to document the set theoretic origin of cognitive difficulties with the cognition of linear algebra concepts.

Our investigations identified three main sources of struggle originating from set theory. This three by all means may not be the only sources of difficulties. The work provided here however may become a spring board for future studies in identifying other sources and providing recommendations for addressing set theory originated learning difficulties, not only in linear algebra but in all mathematics courses where set theory knowledge is an essential prerequisite for a successful mastery of its objects.

### References

- [1] BRENTON, L., and EDWARDS, G. T. Sets of Sets: A Cognitive Obstacle. *The College Mathematics Journal*, 34, n1, 31-38, 2003.
- [2] DOGAN-DUNLAP, H. Visual Instruction of Abstract Concepts for Non-major Students. *The International Journal of Engineering Education (IJEE)*, 20. n4, 671-676, 2003.
- [3] DOGAN-DUNLAP, H. Lack of Set Theory-Relevant Prerequisite Knowledge. *International Journal of Mathematical Education in Science and Technology (IJMEST)*. Vol. 37, No. 4, June 2006. pp. 401-410.

- [4] DOGAN-DUNLAP, H. Linear Algebra Students' Modes of Reasoning: Geometric Representations. *Linear Algebra and Its Applications* (LAA), 432. pp. 2141-2159, 2010.
- [5] DORIER, J., ROBERT, A., ROBINET, J., and ROGALSKI, M. The Obstacle of Formalism in Linear Algebra. *On the Teaching of Linear Algebra* (Kluwer Academic Publishers) Mathematics Education Library, 23, 85-124, 2000.
- [6] DORIER, J., and ROBERT, A. On a Research Programme Concerning the Teaching and Learning of Linear Algebra in the First Year of a French Science University. *International Journal of Mathematics Education in Science & Technology*, 31, Issue 1, 27-35, 2000.
- [7] DORIER, J. and SIERPINSKA, A. Research into the Teaching and Learning of Linear Algebra. In Derek Holton (Ed.) *The Teaching and Learning of Mathematics of University level*. Kluwer Academic Publishers, DorDrecht, 255-273, 2001.

**Received: August, 2011**