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Characterizations and bounds for weighted sums of eigenvalues of normal and Hermitian matrices

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Abstract

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be normal with eigenvalues $\lambda_1, \ldots, \lambda_n$, and let $t_1, \ldots, t_n \in \mathbb{C}$. It is well-known that

$$\max_{\pi\in S_n} |t_1\lambda_{\pi(1)} + \cdots + t_n\lambda_{\pi(n)}| = \max\Big\{|t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n| \;\Big|\; \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset_o \mathbb{C}^n\Big\}.$$

Here S_n denotes the symmetric group of order n, and $\subset_o$ means "is an orthonormal subset of ...". If $\mathbf{A}$ is Hermitian and $\lambda_1 \geq \cdots \geq \lambda_n$, and if $t_1, \ldots, t_n \in \mathbb{R}$ satisfy $t_1 \geq \cdots \geq t_n$, then

$$t_1\lambda_1 + \cdots + t_n\lambda_n = \max\Big\{t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n \mid \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset_o \mathbb{C}^n\Big\}$$

and

$$t_n\lambda_1 + \cdots + t_1\lambda_n = \min\Big\{t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n \mid \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset_o \mathbb{C}^n\Big\}.$$

We present bounds for the left-hand sides of all these equations by suitable choices of $\mathbf{u}_1, \ldots, \mathbf{u}_n$.

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1 Introduction

Let $\mathbf{A} = (a_{ij}) \in \mathbb{C}^{n\times n}$ have eigenvalues $\lambda_1, \dots, \lambda_n$, ordered $\lambda_1 \geq \cdots \geq \lambda_n$ if they are real. We use this notation throughout and assume that $n \geq 2$. We also let $t_1, \dots, t_n$ throughout denote given complex numbers, ordered $t_1 \geq \cdots \geq t_n$ if they are real.

If $\mathbf{A}$ is normal, then

$$\max_{\pi\in S_n} |t_1\lambda_{\pi(1)} + \cdots + t_n\lambda_{\pi(n)}| = \max\Big\{|t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n| \;\Big|\; \{\mathbf{u}_1, \dots, \mathbf{u}_n\} \subset_o \mathbb{C}^n\Big\} \tag{1}$$

(Li, Tam and Tsing [5, Theorem 4.1]). Here S_n denotes the symmetric group of order n, and $\subset_o$ means "is an orthonormal subset of ...". Previously, Mirsky [9, Theorem 6] proved (1) assuming that the t_i's are real. Putting $t_1 = 1$, $t_2 = \cdots = t_{n-1} = 0$, $t_n = -1$, he obtained

$$\max_{1\leq i,j\leq n} |\lambda_i - \lambda_j| = \max\Big\{|\mathbf{u}^*\mathbf{A}\mathbf{u} - \mathbf{v}^*\mathbf{A}\mathbf{v}| \;\Big|\; \{\mathbf{u}, \mathbf{v}\} \subset_o \mathbb{C}^n\Big\}. \tag{2}$$

Johnson, Kumar and Wolkowicz [4] found lower bounds for $\max_{i,j} |\lambda_i - \lambda_j|$ by choosing $\mathbf{u}$ and $\mathbf{v}$ suitably. The present authors [8, Theorem 1] proved that also

$$\max_{1\leq i,j\leq n} |\lambda_i - \lambda_j| = \max\Big\{|\mathbf{u}^*\mathbf{A}\mathbf{u} - \mathbf{v}^*\mathbf{A}\mathbf{v}| \;\Big|\; \mathbf{u}, \mathbf{v} \in \mathbb{C}^n, \|\mathbf{u}\| = \|\mathbf{v}\| = 1\Big\}, \tag{3}$$

where $\|.\|$ denotes the Euclidean norm. Hence they found lower bounds for $\max_{i,j} |\lambda_i - \lambda_j|$. We will in Section 2 continue this study by presenting lower bounds for the left-hand side of (1) where certain t_i's are put zero, by suitable choices of $\mathbf{u}_1, \dots, \mathbf{u}_n$.

Assume now that the t_i's are real. Then (1) strengthens into

$$\operatorname{co}\{t_1\lambda_{\pi(1)} + \cdots + t_n\lambda_{\pi(n)} \mid \pi \in S_n\} = \Big\{t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n \mid \{\mathbf{u}_1, \dots, \mathbf{u}_n\} \subset_o \mathbb{C}^n\Big\} \tag{4}$$

(Marcus, Moyls and Filippenko [6, Theorem 1]). Here co denotes the convex hull. If $\mathbf{A}$ is Hermitian, then the set (4) is a line segment in the real axis. Since its left end point is $t_n\lambda_1 + \cdots + t_1\lambda_n$ and right $t_1\lambda_1 + \cdots + t_n\lambda_n$ (e.g., [1, Theorem 368]), we have

$$t_1\lambda_1 + \cdots + t_n\lambda_n = \max\Big\{t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n \mid \{\mathbf{u}_1, \dots, \mathbf{u}_n\} \subset_o \mathbb{C}^n\Big\}, \tag{5}$$

$$t_n\lambda_1 + \cdots + t_1\lambda_n = \min\Big\{t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n \mid \{\mathbf{u}_1, \dots, \mathbf{u}_n\} \subset_o \mathbb{C}^n\Big\}. \tag{6}$$

Let $1 \leq k \leq n$. Putting $t_1 = \cdots = t_k = 1$, $t_{k+1} = \cdots = t_n = 0$ gives the well-known characterizations

$$\lambda_1 + \cdots + \lambda_k = \max\left\{\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + \mathbf{u}_k^*\mathbf{A}\mathbf{u}_k \mid \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset_o \mathbb{C}^n\right\}, \quad (7)$$

$$\lambda_{n-k+1} + \cdots + \lambda_n = \min\left\{\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + \mathbf{u}_k^*\mathbf{A}\mathbf{u}_k \mid \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subset_o \mathbb{C}^n\right\} \quad (8)$$

(e.g. [2, Corollary 4.3.18]). We will in Section 3 present lower bounds for the left-hand side of (5) and upper bounds for that of (6) when certain t_i's are zero, by suitable choices of $\mathbf{u}_i$'s.

We will complete our paper with examples in Section 4 and computer experiments in Section 5.

2 Studying $\max_{\pi \in S_n} |t_1\lambda_{\pi(1)} + \cdots + t_n\lambda_{\pi(n)}|$, A normal

Throughout this section, we assume that $\mathbf{A} \in \mathbb{C}^{n\times n}$ is normal and $r, s \in \mathbb{C}$.

2.1 The case $t_3 = \cdots = t_n = 0$

Choosing in (1) $t_1 = r$, $t_2 = s$, $t_3 = \cdots = t_n = 0$, we have

$$\max_{\substack{1\leq i,j\leq n\\ i\neq j}} |r\lambda_i + s\lambda_j| = \max\left\{|r\mathbf{u}^*\mathbf{A}\mathbf{u} + s\mathbf{v}^*\mathbf{A}\mathbf{v}| \;\middle|\; \{\mathbf{u}, \mathbf{v}\} \subset_o \mathbb{C}^n\right\}. \quad (9)$$

Let $\operatorname{su}\mathbf{A}$ denote the sum of the entries of $\mathbf{A}$, and denote $a_i = a_{ii}$ $(i = 1, \ldots, n)$. We generalize [8, Theorem 5] to

$$\max_{\substack{1\leq i,j\leq n\\ i\neq j}} |r\lambda_i + s\lambda_j| \geq \max_{\substack{1\leq i,j\leq n\\ i\neq j}} \left|\frac{r}{n}\operatorname{su}\mathbf{A} + \frac{s}{2}(a_i + a_j - a_{ij} - a_{ji})\right| \geq$$
$$\left|\frac{(n-1)r - s}{n(n-1)}\operatorname{su}\mathbf{A} + \frac{s}{n-1}\operatorname{tr}\mathbf{A}\right|. \quad (10)$$

The proof is a straightforward modification of that of [8, Theorem 5] but we present it for completeness. For $i = 1, \ldots, n$, let $\mathbf{e}_i$ be the i'th standard basis vector of $\mathbb{C}^n$, and let $\mathbf{e}$ be the vector of ones. In (9), set $\mathbf{u} = \frac{1}{\sqrt{n}}\mathbf{e}$ and $\mathbf{v} = \frac{1}{\sqrt{2}}(\mathbf{e}_i - \mathbf{e}_j)$ where $i, j = 1, \ldots, n$, $i \neq j$. Then

$$r\mathbf{u}^*\mathbf{A}\mathbf{u} + s\mathbf{v}^*\mathbf{A}\mathbf{v} = \frac{r}{n}\operatorname{su}\mathbf{A} + \frac{s}{2}(a_i + a_j - a_{ij} - a_{ji}),$$

and the first inequality of (10) follows. We underestimate its right-hand side. If $w, z_1, \dots, z_p \in \mathbb{C}$, then clearly

$$\left|w + \frac{z_1 + \cdots + z_p}{p}\right| \le \max_{1\le i\le p} |w + z_i|.$$

Let $w = \frac{r}{n}\operatorname{su}\mathbf{A}$ and let the z_t's be the $n(n-1)$ numbers

$$z_{ij} = \frac{s}{2}(a_i + a_j - a_{ij} - a_{ji}).$$

Then

$$\max_{\substack{1\le i,j\le n\\ i\ne j}} \left|\frac{r}{n}\operatorname{su}\mathbf{A} + \frac{s}{2}(a_i + a_j - a_{ij} - a_{ji})\right| \ge$$

$$\left|\frac{r}{n}\operatorname{su}\mathbf{A} + \frac{s}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^{n} \frac{a_i + a_j - a_{ij} - a_{ji}}{2}\right| =$$

$$\left|\frac{r}{n}\operatorname{su}\mathbf{A} + \frac{s}{n(n-1)}\sum_{i,j=1}^{n} \frac{a_i + a_j - a_{ij} - a_{ji}}{2}\right| =$$

$$\left|\frac{r}{n}\operatorname{su}\mathbf{A} + \frac{s}{n(n-1)}(n\operatorname{tr}\mathbf{A} - \operatorname{su}\mathbf{A})\right| = \left|\frac{(n-1)r - s}{n(n-1)}\operatorname{su}\mathbf{A} + \frac{s}{n-1}\operatorname{tr}\mathbf{A}\right|,$$

which completes the proof.

For $r = 1$, $s = -1$ (or $r = -1$, $s = 1$),

$$\max_{1\le i,j\le n} |\lambda_i - \lambda_j| \ge \frac{|\operatorname{su}\mathbf{A} - \operatorname{tr}\mathbf{A}|}{n-1} = \frac{1}{n-1}\Big|\sum_{\substack{i,j=1\\ i\ne j}}^{n} a_{ij}\Big|, \tag{11}$$

repeating [8, Theorem 5] (and its special case [4, Theorem 2.1]). For $r = s = 1$,

$$\max_{\substack{1\le i,j\le n\\ i\ne j}} |\lambda_i + \lambda_j| \ge \left|\frac{n-2}{n(n-1)}\operatorname{su}\mathbf{A} + \frac{1}{n-1}\operatorname{tr}\mathbf{A}\right|;$$

for $r = 1$, $s = 0$,

$$\max_{1\le i\le n} |\lambda_i| \ge \frac{|\operatorname{su}\mathbf{A}|}{n}$$

(Parker [10, Theorem 3]); and for $r = 0$, $s = 1$,

$$\max_{1\le j\le n} |\lambda_j| \ge \frac{|n\operatorname{tr}\mathbf{A} - \operatorname{su}\mathbf{A}|}{n(n-1)}.$$

2.2 The case $t_{k+1} = \cdots = t_n = 0$

Let us define the following notations and use them throughout: $[n] = \{1, \ldots, n\}$, $\mathbf{e}_I = \sum_{i \in I} \mathbf{e}_i$ where $\emptyset \neq I \subseteq [n]$, $\mathbf{A}_I$ is the principal submatrix of $\mathbf{A}$ with indices in I, and $|I|$ is the number of elements of I.

Johnson, Kumar and Wolkowicz [4, Theorem 2.2(i)] proved that

$$\max_{1 \le i,j \le n} |\lambda_i - \lambda_j| \ge \max_{\substack{\emptyset \neq I, J \subset [n] \\ I \cap J = \emptyset}} \left| \frac{1}{|I|} \operatorname{su} \mathbf{A}_I - \frac{1}{|J|} \operatorname{su} \mathbf{A}_J \right| \ge \max_{1 \le i,j \le n} |a_i - a_j|. \tag{12}$$

Also sets with $I \cap J \neq \emptyset$ can be included, see [8, Theorem 8].

We generalize (12). Let $1 \le k \le n$. Throughout denote $\mathcal{N} = \{\{I_1, \ldots I_p\} \mid 1 \le p \le n$ and $I_1, \ldots, I_p$ are nonempty disjoint subsets of $[n]\}$. We claim that

$$\max_{\pi \in S_n} |t_1 \lambda_{\pi(1)} + \cdots + t_k \lambda_{\pi(k)}| \ge \max_{\{I_1, \ldots, I_k\} \in \mathcal{N}} \left| \frac{t_1}{|I_1|} \operatorname{su} \mathbf{A}_{I_1} + \cdots + \frac{t_k}{|I_k|} \operatorname{su} \mathbf{A}_{I_k} \right|. \tag{13}$$

In particular,

$$\max_{\pi \in S_n} |t_1 \lambda_{\pi(1)} + \cdots + t_k \lambda_{\pi(k)}| \ge \max_{\pi \in S_n} |t_1 a_{\pi(1)} + \cdots + t_k a_{\pi(k)}|. \tag{14}$$

To prove (13), we apply (1) with $t_{k+1} = \cdots = t_n = 0$, $\{I_1, \ldots I_k\} \in \mathcal{N}$, set

$$\mathbf{u}_1 = \frac{1}{\sqrt{|I_1|}} \mathbf{e}_{I_1}, \ldots, \mathbf{u}_k = \frac{1}{\sqrt{|I_k|}} \mathbf{e}_{I_k}, \tag{15}$$

and take the relevant maximum. Restricting $|I_1| = \cdots = |I_k| = 1$ implies (14).

Choosing $t_1 = 1$, $t_2 = -1$, $t_3 = \cdots = t_k = 0$ repeats (12). Choosing $t_1 = 1$, $t_2 = \cdots = t_k = 0$, we obtain

$$\max_{1 \le i \le n} |\lambda_i| \ge \max_{\emptyset \neq I \subseteq [n]} \frac{1}{|I|} |\operatorname{su} \mathbf{A}_I| \ge \max_{1 \le i \le n} |a_i|$$

(Parker [10, Theorem 3]). Finally, choose $t_1 = \cdots = t_k = 1$, $t_{k+1} = \cdots = t_n = 0$ and let $k = 1, \ldots, n-1$. Then, by (14),

$$\begin{gathered}
\max_{1 \le i \le n} |\lambda_i| \ge \max_{1 \le i \le n} |a_i|, \\
\max_{\substack{1 \le i,j \le n \\ i \neq j}} |\lambda_i + \lambda_j| \ge \max_{\substack{1 \le i,j \le n \\ i \neq j}} |a_i + a_j|, \\
\vdots \\
\max_{\substack{1 \le i_1, \ldots, i_{n-1} \le n \\ i_1 \neq \cdots \neq i_{n-1}}} |\lambda_{i_1} + \cdots + \lambda_{i_{n-1}}| \ge \max_{\substack{1 \le i_1, \ldots, i_{n-1} \le n \\ i_1 \neq \cdots \neq i_{n-1}}} |a_{i_1} + \cdots + a_{i_{n-1}}|.
\end{gathered}$$

(Here $i_1 \neq \cdots \neq i_{n-1}$ means that $i_1, \ldots, i_{n-1}$ are all inequal.) Together with $\lambda_1 + \cdots + \lambda_n = a_1 + \cdots + a_n$, this is a reminiscent of the fact that the eigenvalues of a Hermitian matrix majorize its diagonal entries.

3 Studying $t_1\lambda_1 + \cdots + t_n\lambda_n$, A Hermitian

Throughout this section, $\mathbf{A} \in \mathbb{C}^{n\times n}$ is Hermitian and the t_i's are real. Let us first note that setting $k = n$ in (7) and (8) repeats the elementary fact

$$\operatorname{tr}\mathbf{A} = \mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + \mathbf{u}_n^*\mathbf{A}\mathbf{u}_n$$

for all $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subset_o \mathbb{C}^n$.

3.1 The cases $t_3 = \cdots = t_n = 0$ and $t_2 = \cdots = t_{n-1} = 0$

Choosing $t_1 = 1$, $t_2 = \cdots = t_{n-1} = 0$, $t_n = -1$ in (5) (or applying (2)) yields

$$\lambda_1 - \lambda_n = \max\Big\{\mathbf{u}^*\mathbf{A}\mathbf{u} - \mathbf{v}^*\mathbf{A}\mathbf{v} \mid \{\mathbf{u}, \mathbf{v}\} \subset_o \mathbb{C}^n\Big\},$$

while, by (3),

$$\lambda_1 - \lambda_n = \max\Big\{\mathbf{u}^*\mathbf{A}\mathbf{u} - \mathbf{v}^*\mathbf{A}\mathbf{v} \mid \mathbf{u}, \mathbf{v} \in \mathbb{C}^n, \|\mathbf{u}\| = \|\mathbf{v}\| = 1\Big\}. \tag{16}$$

More generally, let $r, s, t \in \mathbb{R}$ satisfy $r \geq s \geq 0$ and $t \geq 0$. We apply (5) and (6) and proceed as in Section 2.1. Then ($t_1 = r$, $t_2 = s$, $t_3 = \cdots = t_n = 0$)

$$r\lambda_1 + s\lambda_2 = \max\Big\{r\mathbf{u}^*\mathbf{A}\mathbf{u} + s\mathbf{v}^*\mathbf{A}\mathbf{v} \mid \{\mathbf{u}, \mathbf{v}\} \subset_o \mathbb{C}^n\Big\},$$
$$s\lambda_{n-1} + r\lambda_n = \min\Big\{r\mathbf{u}^*\mathbf{A}\mathbf{u} + s\mathbf{v}^*\mathbf{A}\mathbf{v} \mid \{\mathbf{u}, \mathbf{v}\} \subset_o \mathbb{C}^n\Big\},$$

and ($t_1 = r$, $t_2 = \cdots = t_{n-1} = 0$, $t_n = -t$)

$$r\lambda_1 - t\lambda_n = \max\Big\{r\mathbf{u}^*\mathbf{A}\mathbf{u} - t\mathbf{v}^*\mathbf{A}\mathbf{v} \mid \{\mathbf{u}, \mathbf{v}\} \subset_o \mathbb{C}^n\Big\}, \tag{17}$$
$$-t\lambda_1 + r\lambda_n = \min\Big\{r\mathbf{u}^*\mathbf{A}\mathbf{u} - t\mathbf{v}^*\mathbf{A}\mathbf{v} \mid \{\mathbf{u}, \mathbf{v}\} \subset_o \mathbb{C}^n\Big\}. \tag{18}$$

Take $\mathbf{u}$ and $\mathbf{v}$ as in the proof of (10). Then

$$r\lambda_1 + s\lambda_2 \geq \frac{r}{n}\operatorname{su}\mathbf{A} + s\max_{\substack{1\leq i,j\leq n\\ i\neq j}}\Big(\frac{a_i + a_j}{2} - \Re\, a_{ij}\Big) \geq$$
$$\frac{(n-1)r - s}{n(n-1)}\operatorname{su}\mathbf{A} + \frac{s}{n-1}\operatorname{tr}\mathbf{A}, \tag{19}$$

$$s\lambda_{n-1} + r\lambda_n \leq \frac{r}{n}\operatorname{su}\mathbf{A} + s\min_{\substack{1\leq i,j\leq n\\ i\neq j}}\Big(\frac{a_i + a_j}{2} - \Re\, a_{ij}\Big) \leq$$
$$\frac{(n-1)r - s}{n(n-1)}\operatorname{su}\mathbf{A} + \frac{s}{n-1}\operatorname{tr}\mathbf{A}, \tag{20}$$

$$r\lambda_1 - t\lambda_n \geq \frac{r}{n}\,\mathrm{su}\,\mathbf{A} - t \min_{\substack{1\leq i,j\leq n\\ i\neq j}} \Big(\frac{a_i+a_j}{2} - \Re\, a_{ij}\Big) \geq \frac{(n-1)r+t}{n(n-1)}\,\mathrm{su}\,\mathbf{A} - \frac{t}{n-1}\,\mathrm{tr}\,\mathbf{A}, \tag{21}$$

$$-t\lambda_1 + r\lambda_n \leq \frac{r}{n}\,\mathrm{su}\,\mathbf{A} - t \max_{\substack{1\leq i,j\leq n\\ i\neq j}} \Big(\frac{a_i+a_j}{2} - \Re\, a_{ij}\Big) \leq \frac{(n-1)r+t}{n(n-1)}\,\mathrm{su}\,\mathbf{A} - \frac{t}{n-1}\,\mathrm{tr}\,\mathbf{A}. \tag{22}$$

Here $\Re$ denotes the real part.

For $r = t = 1$, the second bounds (21) and (22) imply

$$\lambda_1 - \lambda_n \geq \frac{\mathrm{su}\,\mathbf{A} - \mathrm{tr}\,\mathbf{A}}{n-1} = \frac{1}{n-1}\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij} = \frac{2}{n-1}\sum_{\substack{i,j=1\\ i<j}}^{n} \Re\, a_{ij},$$
$$-\lambda_1 + \lambda_n \leq \text{the same expression},$$

and so

$$\lambda_1 - \lambda_n \geq \frac{|\mathrm{su}\,\mathbf{A} - \mathrm{tr}\,\mathbf{A}|}{n-1} = \frac{1}{n-1}\Big|\sum_{\substack{i,j=1\\ i\neq j}}^{n} a_{ij}\Big| = \frac{2}{n-1}\Big|\sum_{\substack{i,j=1\\ i<j}}^{n} \Re\, a_{ij}\Big|, \tag{23}$$

compatibly with (11). The first bounds (21) and (22) improve (23) to

$$\lambda_1 - \lambda_n \geq \max\Big\{\frac{\mathrm{su}\,\mathbf{A}}{n} - \min_{\substack{1\leq i,j\leq n\\ i\neq j}}\Big(\frac{a_i+a_j}{2} - \Re\, a_{ij}\Big),\ -\frac{\mathrm{su}\,\mathbf{A}}{n} + \max_{\substack{1\leq i,j\leq n\\ i\neq j}}\Big(\frac{a_i+a_j}{2} - \Re\, a_{ij}\Big)\Big\}. \tag{24}$$

3.2 The cases $t_{k+1} = \cdots = t_n = 0$ and $t_{k+1} = \cdots = t_{n-l} = 0$

Let $1 \leq k \leq n$. Applying (5) and (6) with $t_{k+1} = \cdots = t_n = 0$ and proceeding as in the proof of (13) yields

$$t_1\lambda_1 + \cdots + t_k\lambda_k \geq \max_{\{I_1,\ldots,I_k\}\in\mathcal{N}} \Big(\frac{t_1}{|I_1|}\mathrm{su}\,\mathbf{A}_{I_1} + \cdots + \frac{t_k}{|I_k|}\mathrm{su}\,\mathbf{A}_{I_k}\Big), \tag{25}$$

$$t_k\lambda_{n-k+1} + \cdots + t_1\lambda_n \leq \min_{\{I_1,\ldots,I_k\}\in\mathcal{N}} \Big(\frac{t_1}{|I_1|}\mathrm{su}\,\mathbf{A}_{I_1} + \cdots + \frac{t_k}{|I_k|}\mathrm{su}\,\mathbf{A}_{I_k}\Big). \tag{26}$$

Let $a_{[1]} \geq \cdots \geq a_{[n]}$ be the ordered diagonal entries of $\mathbf{A}$. Restricting $|I_1| = \cdots = |I_k| = 1$, we have

$$t_1\lambda_1 + \cdots + t_k\lambda_k \geq t_1 a_{[1]} + \cdots + t_k a_{[k]},$$
$$t_k\lambda_{n-k+1} + \cdots + t_1\lambda_n \leq t_1 a_{[n-k+1]} + \cdots + t_k a_{[n]},$$

and in particular

$$\lambda_1 + \cdots + \lambda_k \geq a_{[1]} + \cdots + a_{[k]}, \qquad (27)$$
$$\lambda_{n-k+1} + \cdots + \lambda_n \leq a_{[n-k+1]} + \cdots + a_{[n]}.$$

Inequalities (27) where $k = 1, \dots, n-1$, together with $\lambda_1 + \cdots + \lambda_n = a_1 + \cdots + a_n$, tell the well-known fact (e.g., [2, Theorem 4.3.26], [7, Section 9.B.1]) that the vector $(\lambda_1, \dots, \lambda_n)$ majorizes the vector $(a_1, \dots, a_n)$.

Setting $k = t_1 = 1$ in (25) and (26) gives the well-known inequalities

$$\lambda_1 \geq \max_{\emptyset \neq I \subseteq [n]} \frac{1}{|I|} \operatorname{su} \mathbf{A}_I \geq a_{[1]}, \qquad \lambda_n \leq \min_{\emptyset \neq I \subseteq [n]} \frac{1}{|I|} \operatorname{su} \mathbf{A}_I \leq a_{[n]}.$$

If $k + l \leq n$ and $t_1 \geq \cdots \geq t_k \geq 0 = t_{k+1} = \cdots = t_{n-l} \geq -t_{n-l+1} \geq \cdots \geq -t_n$, then, by (5) and (6),

$$t_1\lambda_1 + \cdots + t_k\lambda_k - t_{n-l+1}\lambda_{n-l+1} - \cdots - t_n\lambda_n =$$
$$\max\Big\{ t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_k\mathbf{u}_k^*\mathbf{A}\mathbf{u}_k - t_{n-l+1}\mathbf{u}_{n-l+1}^*\mathbf{A}\mathbf{u}_{n-l+1} - \cdots -$$
$$t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n \,\Big|\, \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{n-l+1}, \dots, \mathbf{u}_n\} \subset_o \mathbb{C}^n \Big\} \qquad (28)$$

and

$$-t_n\lambda_1 - \cdots - t_{n-l+1}\lambda_l + t_k\lambda_{n-k+1} + \cdots + t_1\lambda_n =$$
$$\min\Big\{ t_1\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1 + \cdots + t_k\mathbf{u}_k^*\mathbf{A}\mathbf{u}_k - t_{n-l+1}\mathbf{u}_{n-l+1}^*\mathbf{A}\mathbf{u}_{n-l+1} - \cdots -$$
$$t_n\mathbf{u}_n^*\mathbf{A}\mathbf{u}_n \,\Big|\, \{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{u}_{n-l+1}, \dots, \mathbf{u}_n\} \subset_o \mathbb{C}^n \Big\}. \qquad (29)$$

The case $k = l = 1$ repeats (17) and (18).

Let $\{I_1, \dots, I_k, I_{n-l+1}, \dots, I_n\} \in \mathcal{N}$. We choose

$$\mathbf{u}_1 = \frac{1}{\sqrt{|I_1|}}\mathbf{e}_{I_1}, \dots, \mathbf{u}_k = \frac{1}{\sqrt{|I_k|}}\mathbf{e}_{I_k},$$
$$\mathbf{u}_{n-l+1} = \frac{1}{\sqrt{|I_{n-l+1}|}}\mathbf{e}_{I_{n-l+1}}, \dots, \mathbf{u}_n = \frac{1}{\sqrt{|I_n|}}\mathbf{e}_{I_n}.$$

Applying (28) and taking the relevant maximum yields

$$t_1\lambda_1 + \cdots + t_k\lambda_k - t_{n-l+1}\lambda_{n-l+1} - \cdots - t_n\lambda_n \geq$$
$$\max_{\{I_1, \dots, I_k, I_{n-l+1}, \dots, I_n\} \in \mathcal{N}} \Big(\frac{t_1}{|I_1|}\operatorname{su}\mathbf{A}_{I_1} + \cdots + \frac{t_k}{|I_k|}\operatorname{su}\mathbf{A}_{I_k} -$$
$$\frac{t_{n-l+1}}{|I_{n-l+1}|}\operatorname{su}\mathbf{A}_{I_{n-l+1}} - \cdots - \frac{t_n}{|I_n|}\operatorname{su}\mathbf{A}_{I_n}\Big). \qquad (30)$$

Hence, restricting $|I_1| = \cdots = |I_k| = |I_{n-l+1}| = \cdots = |I_n| = 1$,

$$t_1\lambda_1 + \cdots + t_k\lambda_k - t_{n-l+1}\lambda_{n-l+1} - \cdots - t_n\lambda_n \geq \\ t_1 a_{[1]} + \cdots + t_k a_{[k]} - t_n a_{[n-l+1]} - \cdots - t_{n-l+1} a_{[n]}.$$

In particular,

$$\lambda_1 - \lambda_n \geq \max_{\substack{\emptyset \neq I, J \subset [n] \\ I \cap J = \emptyset}} \left(\frac{1}{|I|} \operatorname{su} \mathbf{A}_I - \frac{1}{|J|} \operatorname{su} \mathbf{A}_J \right) \geq a_{[1]} - a_{[n]},$$

but (16) (or [8, Theorem 8(i)]) gives stronger

$$\lambda_1 - \lambda_n \geq \max_{\emptyset \neq I, J \subset [n]} \left(\frac{1}{|I|} \operatorname{su} \mathbf{A}_I - \frac{1}{|J|} \operatorname{su} \mathbf{A}_J \right) \geq a_{[1]} - a_{[n]}. \tag{31}$$

Similarly, by (29),

$$-t_n\lambda_1 - \cdots - t_{n-l+1}\lambda_l + t_k\lambda_{n-k+1} + \cdots + t_1\lambda_n \leq \\ \min_{\{I_1,\ldots,I_k,I_{n-l+1},\ldots,I_n\} \in \mathcal{N}} \left(\frac{t_1}{|I_1|} \operatorname{su} \mathbf{A}_{I_1} + \cdots + \frac{t_k}{|I_k|} \operatorname{su} \mathbf{A}_{I_k} - \right. \\ \left. \frac{t_{n-l+1}}{|I_{n-l+1}|} \operatorname{su} \mathbf{A}_{I_{n-l+1}} - \cdots - \frac{t_n}{|I_n|} \operatorname{su} \mathbf{A}_{I_n} \right), \tag{32}$$

and, restricting as above,

$$-t_n\lambda_1 - \cdots - t_{n-l+1}\lambda_l + t_k\lambda_{n-k+1} + \cdots + t_1\lambda_n \leq \\ t_k a_{[1]} + \cdots + t_1 a_{[k]} - t_{n-l+1} a_{[n-l+1]} - \cdots - t_n a_{[n]}.$$

3.3 Restricting $|I_1| = \cdots = |I_k| = 2$

Generalizing (15), set

$$\mathbf{u}_1 = \frac{1}{\sqrt{|I_1|}} \sum_{i \in I_1} \mathrm{e}^{\mathrm{i}\theta_i} \mathbf{e}_i, \ldots, \mathbf{u}_k = \frac{1}{\sqrt{|I_k|}} \sum_{i \in I_k} \mathrm{e}^{\mathrm{i}\theta_i} \mathbf{e}_i,$$

where the θ_i's are arbitrary real numbers. To maximize and minimize $\mathbf{u}_1^*\mathbf{A}\mathbf{u}_1, \ldots, \mathbf{u}_k^*\mathbf{A}\mathbf{u}_k$ over the θ_i's is difficult in general but easy if $|I_1| = \cdots = |I_k| = 2$.

Under this assumption, let $1 \leq i, j \leq n$, $i \neq j$, $\mathbf{u}_{ij}(\theta, \phi) = \frac{1}{\sqrt{2}}(\mathrm{e}^{\mathrm{i}\theta}\mathbf{e}_i + \mathrm{e}^{\mathrm{i}\phi}\mathbf{e}_j)$ and $a_{ij} = |a_{ij}|\mathrm{e}^{\mathrm{i}\alpha_{ij}}$. The maximum of

$$\mathbf{u}_{ij}^*\mathbf{A}\mathbf{u}_{ij} = \frac{a_i + a_j}{2} + \Re(\mathrm{e}^{\mathrm{i}(\phi-\theta)} a_{ij}) \tag{33}$$

is

$$\frac{a_i + a_j}{2} + |a_{ij}|,$$

attained for

$$\mathbf{u}_{ij}(0, -\alpha_{ij}) = \mathbf{v}_{ij}.$$

The minimum of (33) is

$$\frac{a_i + a_j}{2} - |a_{ij}|,$$

attained for

$$\mathbf{u}_{ij}(0, -\alpha_{ij} + \pi) = \mathbf{w}_{ij}.$$

Now let $1 \le k \le \frac{n}{2}$, $t_1 \ge \cdots \ge t_k \ge 0 = t_{k+1} = \cdots = t_n$, $1 \le i_1, j_1, \ldots, i_k, j_k \le n$, $i_1 \ne j_1 \ne \cdots \ne i_k \ne j_k$, $I_1 = \{i_1, j_1\}, \ldots, I_k = \{i_k, j_k\}$. By (5) and (6),

$$t_1\lambda_1 + \cdots + t_k\lambda_k \ge t_1\mathbf{v}_{i_1j_1}^*\mathbf{A}\mathbf{v}_{i_1j_1} + \cdots + t_k\mathbf{v}_{i_kj_k}^*\mathbf{A}\mathbf{v}_{i_kj_k} = \\ t_1\frac{a_{i_1} + a_{j_1}}{2} + \cdots + t_k\frac{a_{i_k} + a_{j_k}}{2} + t_1|a_{i_1j_1}| + \cdots + t_k|a_{i_kj_k}|$$

and

$$t_k\lambda_{n-k+1} + \cdots + t_1\lambda_n \le t_1\mathbf{w}_{i_1j_1}^*\mathbf{A}\mathbf{w}_{i_1j_1} + \cdots + t_k\mathbf{w}_{i_kj_k}^*\mathbf{A}\mathbf{w}_{i_kj_k} = \\ t_1\frac{a_{i_1} + a_{j_1}}{2} + \cdots + t_k\frac{a_{i_k} + a_{j_k}}{2} - t_1|a_{i_1j_1}| - \cdots - t_k|a_{i_kj_k}|.$$

Furthermore,

$$t_1\lambda_1 + \cdots + t_k\lambda_k \ge \\ \max_{\substack{i_1 \ne \cdots \ne i_k \ne \\ j_1 \ne \cdots \ne j_k}} \Big(t_1\frac{a_{i_1} + a_{j_1}}{2} + \cdots + t_k\frac{a_{i_k} + a_{j_k}}{2} + t_1|a_{i_1j_1}| + \cdots + t_k|a_{i_kj_k}|\Big) \tag{34}$$

and

$$t_k\lambda_{n-k+1} + \cdots + t_1\lambda_n \le \\ \min_{\substack{i_1 \ne \cdots \ne i_k \ne \\ j_1 \ne \cdots \ne j_k}} \Big(t_1\frac{a_{i_1} + a_{j_1}}{2} + \cdots + t_k\frac{a_{i_k} + a_{j_k}}{2} - t_1|a_{i_1j_1}| - \cdots - t_k|a_{i_kj_k}|\Big). \tag{35}$$

If n is even and $t_1 = \cdots = t_{\frac{n}{2}} = 1$, $t_{\frac{n}{2}+1} = \cdots = t_n = 0$, then (34) and (35) simplify into

$$\lambda_1 + \cdots + \lambda_{\frac{n}{2}} \ge \frac{\operatorname{tr}\mathbf{A}}{2} + \max_{\substack{i_1 \ne \cdots \ne i_{\frac{n}{2}} \ne \\ j_1 \ne \cdots \ne j_{\frac{n}{2}}}} (|a_{i_1j_1}| + \cdots + |a_{i_{\frac{n}{2}}j_{\frac{n}{2}}}|), \tag{36}$$

$$\lambda_{\frac{n}{2}+1} + \cdots + \lambda_n \le \frac{\operatorname{tr}\mathbf{A}}{2} - \max_{\substack{i_1 \ne \cdots \ne i_{\frac{n}{2}} \ne \\ j_1 \ne \cdots \ne j_{\frac{n}{2}}}} (|a_{i_1j_1}| + \cdots + |a_{i_{\frac{n}{2}}j_{\frac{n}{2}}}|). \tag{37}$$

If n is odd and $t_1 = \cdots = t_{\frac{n+1}{2}} = 1$, $t_{\frac{n+3}{2}} = \cdots = t_n = 0$, then choose $I_1, \ldots I_{\frac{n-1}{2}}$ as above and let $I_{\frac{n+1}{2}}$ be the remaining $\{i\}$. We have

$$\mathbf{v}^*_{i_1j_1}\mathbf{A}\mathbf{v}_{i_1j_1} + \cdots + \mathbf{v}^*_{i_{\frac{n-1}{2}}j_{\frac{n-1}{2}}}\mathbf{A}\mathbf{v}_{i_{\frac{n-1}{2}}j_{\frac{n-1}{2}}} + \mathbf{e}^*_{I_{\frac{n+1}{2}}}\mathbf{A}\mathbf{e}_{I_{\frac{n+1}{2}}} =$$
$$\tfrac{1}{2}a_{i_1} + \tfrac{1}{2}a_{j_1} + \cdots + \tfrac{1}{2}a_{i_{\frac{n-1}{2}}} + \tfrac{1}{2}a_{j_{\frac{n-1}{2}}} + |a_{i_1j_1}| + \cdots + |a_{i_{\frac{n-1}{2}}j_{\frac{n-1}{2}}}| + a_{i_{\frac{n+1}{2}}} =$$
$$\frac{\operatorname{tr}\mathbf{A}}{2} + |a_{i_1j_1}| + \cdots + |a_{i_{\frac{n-1}{2}}j_{\frac{n-1}{2}}}| + \tfrac{1}{2}a_{i_{\frac{n+1}{2}}},$$

and so

$$\lambda_1 + \cdots + \lambda_{\frac{n+1}{2}} \geq \frac{\operatorname{tr}\mathbf{A}}{2} + \max_{\substack{i_1\neq\cdots\neq i_{\frac{n+1}{2}}\neq \\ j_1\neq\cdots\neq j_{\frac{n-1}{2}}}} (|a_{i_1j_1}| + \cdots + |a_{i_{\frac{n-1}{2}}j_{\frac{n-1}{2}}}| + \tfrac{1}{2}a_{i_{\frac{n+1}{2}}}), \quad (38)$$

$$\lambda_{\frac{n+1}{2}} + \cdots + \lambda_n \leq \frac{\operatorname{tr}\mathbf{A}}{2} - \max_{\substack{i_1\neq\cdots\neq i_{\frac{n+1}{2}}\neq \\ j_1\neq\cdots\neq j_{\frac{n-1}{2}}}} (|a_{i_1j_1}| + \cdots + |a_{i_{\frac{n-1}{2}}j_{\frac{n-1}{2}}}| + \tfrac{1}{2}a_{i_{\frac{n+1}{2}}}). \quad (39)$$

By (36) and (37),

$$(\lambda_1 + \cdots + \lambda_{\frac{n}{2}}) - (\lambda_{\frac{n}{2}+1} + \cdots + \lambda_n) \geq 2 \max_{\substack{i_1\neq\cdots\neq i_{\frac{n}{2}}\neq \\ j_1\neq\cdots\neq j_{\frac{n}{2}}}} (|a_{i_1j_1}| + \cdots + |a_{i_{\frac{n}{2}}j_{\frac{n}{2}}}|)$$

if n is even. By (38) and (39),

$$(\lambda_1 + \cdots + \lambda_{\frac{n-1}{2}}) - (\lambda_{\frac{n+3}{2}} + \cdots + \lambda_n) \geq$$
$$\max_{\substack{i_1\neq\cdots\neq i_{\frac{n+1}{2}}\neq \\ j_1\neq\cdots\neq j_{\frac{n-1}{2}}}} \left[2\left(|a_{i_1j_1}| + \cdots + |a_{i_{\frac{n-1}{2}}j_{\frac{n-1}{2}}}|\right) + a_{i_{\frac{n+1}{2}}}\right]$$

if n is odd.

To study analogously the latter part of Section 3.2, let $n \geq 4$, $k, l \geq 1$, $k+l \leq \frac{n}{2}$, $t_1 \geq \cdots \geq t_k \geq 0 \geq -t_{k+1} \geq \cdots \geq -t_{k+l}$, $i_1 \neq j_1 \neq \cdots \neq i_{k+l} \neq j_{k+l}$, $I_1 = \{i_1, j_1\}, \ldots, I_{k+l} = \{i_{k+l}, j_{k+l}\}$. Then, by (5),

$$t_1\lambda_1 + \cdots + t_k\lambda_k - t_{k+1}\lambda_{n-l+1} - \cdots - t_{k+l}\lambda_n \geq$$
$$t_1\mathbf{v}^*_{i_1j_1}\mathbf{A}\mathbf{v}_{i_1j_1} + \cdots + t_k\mathbf{v}^*_{i_kj_k}\mathbf{A}\mathbf{v}_{i_kj_k} -$$
$$t_{k+1}\mathbf{w}^*_{i_{k+1}j_{k+1}}\mathbf{A}\mathbf{w}_{i_{k+1}j_{k+1}} - \cdots - t_{k+l}\mathbf{w}^*_{i_{k+l}j_{k+l}}\mathbf{A}\mathbf{w}_{i_{k+l}j_{k+l}}.$$

Consequently,

$$t_1\lambda_1 + \cdots + t_k\lambda_k - t_{k+1}\lambda_{n-l+1} - \cdots - t_{k+l}\lambda_n \geq$$
$$\max_{\substack{i_1\neq\cdots\neq i_{k+l}\neq \\ j_1\neq\cdots\neq j_{k+l}}} \Big(t_1\frac{a_{i_1} + a_{j_1}}{2} + \cdots + t_k\frac{a_{i_k} + a_{j_k}}{2} - t_{k+1}\frac{a_{i_{k+1}} + a_{j_{k+1}}}{2} - \ldots$$
$$-t_{k+l}\frac{a_{i_{k+l}} + a_{j_{k+l}}}{2} + t_1|a_{i_1j_1}| + \cdots + t_{k+l}|a_{i_{k+l}j_{k+l}}|\Big) \quad (40)$$

and

$$-t_{k+l}\lambda_1 - \cdots - t_{k+1}\lambda_l + t_k\lambda_{n-k+1} + \cdots + t_1\lambda_n \le \min_{\substack{i_1\ne\cdots\ne i_{k+l}\ne\\ j_1\ne\cdots\ne j_{k+l}}} \Big(t_1\frac{a_{i_1}+a_{j_1}}{2} + \cdots + t_k\frac{a_{i_k}+a_{j_k}}{2} - t_{k+1}\frac{a_{i_{k+1}}+a_{j_{k+1}}}{2} - \cdots - t_{k+l}\frac{a_{i_{k+l}}+a_{j_{k+l}}}{2} - t_1|a_{i_1j_1}| - \cdots - t_{k+l}|a_{i_{k+l}j_{k+l}}|\Big).$$

In particular,

$$\lambda_1 - \lambda_n \ge \max_{i\ne j\ne r\ne s}\Big(\frac{a_i+a_j}{2} - \frac{a_r+a_s}{2} + |a_{ij}| + |a_{rs}|\Big), \tag{41}$$

somewhat resembling the bound

$$\lambda_1 - \lambda_n \ge 2\max_{i\ne j}|a_{ij}| \tag{42}$$

(Parker [10, Theorem 7]).

4 Examples

Example 1. Consider the symmetric matrix

$$\mathbf{A} = \begin{pmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{pmatrix},$$

cited from [11, Example 4]. Its eigenvalues are $\lambda_1 = 9.3759$, $\lambda_2 = 6.4230$, $\lambda_3 = 4.7754$, $\lambda_4 = 1.4257$.

Bounds from Section 3.1. We have $\operatorname{su}\mathbf{A} = 34$, $\operatorname{tr}\mathbf{A} = 22$,

$$\max_{i\ne j}\Big(\frac{a_i+a_j}{2} - a_{ij}\Big) = \frac{a_3+a_4}{2} - a_{34} = \tfrac{13}{2}$$

and

$$\min_{i\ne j}\Big(\frac{a_i+a_j}{2} - a_{ij}\Big) = \frac{a_1+a_4}{2} - a_{14} = \tfrac{5}{2}.$$

The first bounds (19) with $r = s = 1$ and respectively $r = 3$, $s = 2$,

$$\begin{aligned} \lambda_1 + \lambda_2 &\ge \tfrac{1}{4}\cdot 34 + \tfrac{13}{2} = 15, \\ 3\lambda_1 + 2\lambda_2 &\ge \tfrac{3}{4}\cdot 34 + 2\cdot\tfrac{13}{2} = 38\tfrac{1}{2}, \end{aligned} \tag{43}$$

are quite good, since actually $\lambda_1 + \lambda_2 = 15.799$, $3\lambda_1 + 2\lambda_2 = 40.920$. The second bounds,

$$\lambda_1 + \lambda_2 \geq \tfrac{1}{6} \cdot 34 + \tfrac{1}{3} \cdot 22 = 13, \quad 3\lambda_1 + 2\lambda_2 \geq \tfrac{7}{12} \cdot 34 + \tfrac{2}{3} \cdot 22 = 34\tfrac{1}{2},$$

are easier to compute but not so good.

The first bound (20) with $r = s = 1$ gives $\lambda_3 + \lambda_4 \leq \frac{1}{4} \cdot 34 + \frac{5}{2} = 11$, poorly. Actually $\lambda_3 + \lambda_4 = 6.201$, and it is in fact trivial that

$$\lambda_3 + \lambda_4 \leq \frac{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4}{2} = \frac{22}{2} = 11.$$

But using $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 22$ and (43), we can do better

$$\lambda_3 + \lambda_4 = 22 - (\lambda_1 + \lambda_2) \leq 22 - 15 = 7. \tag{44}$$

The bound (20) is in this example poor also with other r, s. For instance, $2\lambda_3 + 3\lambda_4 \leq \frac{3}{4} \cdot 34 + 2 \cdot \frac{5}{2} = 30\frac{1}{2}$ while actually $2\lambda_3 + 3\lambda_4 = 13.828$.

By (24),

$$\lambda_1 - \lambda_4 \geq \max\{\tfrac{1}{4} \cdot 34 - \tfrac{5}{2}, -\tfrac{1}{4} \cdot 34 + \tfrac{13}{2}\} = 6. \tag{45}$$

Actually $\lambda_1 - \lambda_4 = 7.950$, and so this bound is satisfactory. The simpler bound (23), $\lambda_1 - \lambda_4 \geq \frac{2}{3} \cdot 6 = 4$, is poor. The first bound (21) with $r = 3, t = 2$,

$$3\lambda_1 - 2\lambda_4 \geq \tfrac{3}{4} \cdot 34 - 2 \cdot \tfrac{5}{2} = 20\tfrac{1}{2},$$

manages rather well but the second bound $3\lambda_1 - 2\lambda_4 \geq \frac{11}{12} \cdot 34 - \frac{2}{3} \cdot 22 = 16\frac{1}{2}$ does not. Actually $3\lambda_1 - 2\lambda_4 = 25.276$. Neither does (22) with $r = 2, t = 3$ succeed. We have $-3\lambda_1 + 2\lambda_4 \leq \frac{2}{4} \cdot 34 - 3 \cdot \frac{13}{2} = -2\frac{1}{2}$, and so $3\lambda_1 - 2\lambda_4 \geq 2\frac{1}{2}$, very poorly.

Bounds from Section 3.2. Set $k = 2$ and $t_1 = t_2 = 1$ and also $t_1 = 3, t_2 = 2$ in (25). In both cases $I_1 = \{1, 4\}$, $I_2 = \{3\}$ is optimal. We have

$$\mathbf{A}_{I_1} = \begin{pmatrix} 4 & 3 \\ 3 & 7 \end{pmatrix}, \qquad \mathbf{A}_{I_2} = (6),$$

and so

$$\lambda_1 + \lambda_2 \geq \tfrac{1}{2} \cdot 17 + 6 = 14\tfrac{1}{2}, \tag{46}$$

$$3\lambda_1 + 2\lambda_2 \geq \tfrac{3}{2} \cdot 17 + 2 \cdot 6 = 37\tfrac{1}{2}, \tag{47}$$

quite well. Instead, (26) has no success. In both cases, $I_1 = \{1\}$, $I_2 = \{2\}$ is optimal. Then $\mathbf{A}_{I_1} = (4)$, $\mathbf{A}_{I_2} = (5)$, and so

$$\lambda_3 + \lambda_4 \leq 4 + 5 = 9, \tag{48}$$

$$2\lambda_3 + 3\lambda_4 \leq 3 \cdot 4 + 2 \cdot 5 = 22, \tag{49}$$

rather poorly. The same trick as in (44) improves

$$\lambda_3 + \lambda_4 = 22 - (\lambda_1 + \lambda_2) \le 22 - 14\tfrac{1}{2} = 7\tfrac{1}{2}. \tag{50}$$

The first bound (31) is attained for $I = \{1, 3, 4\}$, $J = \{1\}$. Then

$$\mathbf{A}_I = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 6 & 0 \\ 3 & 0 & 7 \end{pmatrix}, \qquad \mathbf{A}_J = (4),$$

and so

$$\lambda_1 - \lambda_4 \ge \tfrac{1}{3} \cdot 27 - 4 = 5. \tag{51}$$

The second bound, $\lambda_1 - \lambda_4 \ge 7 - 4 = 3$, is simple but poor. Let us also set $k = l = 1$, $t_1 = 3$, $t_4 = 2$ in (30). We have

$$3\lambda_1 - 2\lambda_4 \ge 27 - 2 \cdot 5 = 17 \tag{52}$$

(optimal $I_1 = \{1, 3, 4\}$, $I_2 = \{2\}$), not well. Neither does (32) succeed. We have $-2\lambda_1 + 3\lambda_4 \le 3 \cdot 5 - \frac{2}{3} \cdot 27 = -3$ (optimal $I_1 = \{2\}$, $I_2 = \{1, 3, 4\}$), and actually $-2\lambda_1 + 3\lambda_4 = -14.475$.

Bounds from Section 3.3. The disadvantage of these bounds, compared with those from Section 3.2, is that only the I_t's with two elements are considered, but the advantage is that these sets are handled more effectively. Thus the bounds from Section 3.3 may or may not improve those from Section 3.2. We look what happens in our example. Since $\mathbf{A}$ is nonnegative, the bound (34) cannot improve (25) if $t_k \ge 0$, and so we skip it. By (35) (or (37) for (53)),

$$\lambda_3 + \lambda_4 \le \tfrac{1}{2} \cdot 10 + \tfrac{1}{2} \cdot 12 - 2 - 1 = 8 \tag{53}$$

(optimal $i_1 = 1, j_1 = 3, i_2 = 2, j_2 = 4$ or $i_1 = 1, j_1 = 4, i_2 = 2, j_2 = 3$) and

$$2\lambda_3 + 3\lambda_4 \le \tfrac{3}{2} \cdot 11 + 11 - 3 \cdot 3 = 18\tfrac{1}{2}$$

(optimal $i_1 = 1, j_1 = 4, i_2 = 2, j_2 = 3$), improving (48) and (49). On the other hand, the bounds by (40) (or (41) for (54)),

$$\begin{gathered} \lambda_1 - \lambda_4 \ge \tfrac{1}{2} \cdot 12 - \tfrac{1}{2} \cdot 10 + 2 + 1 = 4, \\ 3\lambda_1 - 2\lambda_4 \ge \tfrac{3}{2} \cdot 12 - 10 + 3 \cdot 1 + 2 \cdot 2 = 15 \end{gathered} \tag{54}$$

(optimal $i_1 = 2, j_1 = 4, i_2 = 1, j_2 = 3$), do not improve (51) and (52).

Comparison. We compare some of our results with

$$\lambda_1 + \lambda_2 \ge 2\Big(\mu + \frac{\sigma}{\sqrt{n-1}}\Big), \tag{55}$$

$$\lambda_{n-1} + \lambda_n \le 2\Big(\mu - \frac{\sigma}{\sqrt{n-1}}\Big), \tag{56}$$

$$\lambda_1 - \lambda_n \ge 2\sigma, \tag{57}$$

due to Wolkowicz and Styan [11, Theorems 2.3 and 2.5], called "WS bounds". Here

$$\mu = \frac{\operatorname{tr}\mathbf{A}}{n}, \qquad \sigma^2 = \frac{1}{n}\Big[\operatorname{tr}\mathbf{A}^2 - \frac{(\operatorname{tr}\mathbf{A})^2}{n}\Big].$$

We also include Parker's bound (42) in comparison.

Since

$$\mu = \tfrac{11}{2}, \qquad \sigma^2 = \tfrac{1}{4}\big(154 - \tfrac{1}{4}\cdot 484\big) = \tfrac{33}{4},$$

the WS bound (55) gives

$$\lambda_1 + \lambda_2 \geq 2\big(\tfrac{11}{2} + \tfrac{1}{2}\sqrt{11}\big) = 11 + \sqrt{11} = 14.317.$$

Our bounds (43) and (46) are (slightly) better (but require more bookkeeping). The WS bound (56) is

$$\lambda_3 + \lambda_4 \leq 11 - \sqrt{11} = 7.683,$$

and so (44) and (50) are better but (53) is worse. The WS bound (57),

$$\lambda_1 - \lambda_4 \geq 2\cdot\tfrac{1}{2}\sqrt{33} = \sqrt{33} = 5.745,$$

beats (51) but loses to (45). Parker's bound (42),

$$\lambda_1 - \lambda_4 \geq 2\cdot 3 = 6,$$

is as good as (45).

Example 2. The reason why some of our bounds in Example 1 are fairly good is that $\mathbf{A}$ is nonnegative. But if $\mathbf{A}$ has entries with both positive and negative real parts, then, due to cancellation in summing, the bounds are expected to become weaker. (Possible imaginary parts always cancel.) Consider the Hermitian matrix

$$\mathbf{A} = \begin{pmatrix} 4 & -5+\mathrm{i} & 2 & 3 \\ -5-\mathrm{i} & 5 & -4 & 1+2\mathrm{i} \\ 2 & -4 & 6 & -3 \\ 3 & 1-2\mathrm{i} & -3 & 7 \end{pmatrix},$$

obtained from Example 1 by replacing the zero entries with certain negative numbers or complex number with negative real part. Then $\lambda_1 = 12.9723$, $\lambda_2 = 9.3791$, $\lambda_3 = 1.7850$, $\lambda_4 = -2.1365$.

Bounds from Sections 3.1 and 3.2. We have $\operatorname{su}\mathbf{A} = 10$, $\operatorname{tr}\mathbf{A} = 22$,

$$\max_{i\neq j}\Big(\frac{a_i + a_j}{2} - \Re\, a_{ij}\Big) = \frac{a_1 + a_2}{2} - \Re\, a_{12} = \frac{a_2 + a_3}{2} - a_{23} = \frac{a_3 + a_4}{2} - a_{34} = 9\tfrac{1}{2}$$

and

$$\min_{i\neq j}\Big(\frac{a_i+a_j}{2}-\Re\, a_{ij}\Big)=\frac{a_1+a_4}{2}-a_{14}=2\tfrac{1}{2}.$$

By the first bounds (19) and (20),

$$\lambda_1+\lambda_2\geq \tfrac{1}{4}\cdot 10+9\tfrac{1}{2}=12,\qquad \lambda_3+\lambda_4\leq \tfrac{1}{4}\cdot 10+2\tfrac{1}{2}=5,$$

and, by (24),

$$\lambda_1-\lambda_4\geq \max\{\tfrac{1}{4}\cdot 10-2\tfrac{1}{2},\ -\tfrac{1}{4}\cdot 10+9\tfrac{1}{2}\}=7.$$

Actually $\lambda_1+\lambda_2=22.351$, $\lambda_3+\lambda_4=-0.3515$, $\lambda_1-\lambda_4=15.109$, and so these bounds are poor. Also the bounds obtained from Section 3.2 appear to be poor.

Bounds from Section 3.3. Since

$$\max_{i\neq j\neq r\neq s}(|a_{ij}|+|a_{rs}|)=|a_{12}|+|a_{34}|=\sqrt{26}+3,$$

we have by (36) and (37)

$$\lambda_1+\lambda_2\geq 14+\sqrt{26}=19.099, \tag{58}$$

$$\lambda_3+\lambda_4\leq 8-\sqrt{26}=2.901. \tag{59}$$

Furthermore,

$$\max_{i\neq j\neq r\neq s}\Big(\frac{a_i+a_j}{2}-\frac{a_r+a_s}{2}+|a_{ij}|+|a_{rs}|\Big)=$$
$$\frac{a_3+a_4}{2}-\frac{a_1+a_2}{2}+|a_{34}|+|a_{12}|=5+\sqrt{26}=10.099,$$

and so, by (41),

$$\lambda_1-\lambda_4\geq 10.099, \tag{60}$$

which loses slightly to Parker's bound (42),

$$\lambda_1-\lambda_4\geq 2\sqrt{26}=10.198.$$

The bounds (58) and (60) are not bad. Regarding the relative error, the bound (59) is very bad, but its absolute error is of the same magnitude as that of (58) and (60).

The WS bounds (55) and (56),

$$\lambda_1+\lambda_2\geq 11+\sqrt{\frac{143}{3}}=17.904,\quad \lambda_3+\lambda_4\leq 11-\sqrt{\frac{143}{3}}=4.096,$$

lose to (58). On the other hand, the WS bound (57),

$$\lambda_1-\lambda_4\geq\sqrt{143}=11.958,$$

beats (60).

5 Computer experiments

We studied positive symmetric ("PS" in the sequel), real symmetric ("RS"), and complex Hermitian ("CH") 4×4 matrices experimentally. We generated 100 random matrices of each type by using the Matlab generator rand for PS and randn for RS and CH. We set $k = 2$, $r = t_1 = 3$, $s = t_2 = 2$, $t = 1$. For each type and each bound, we computed the mean μ and standard deviation σ of the relative error

$$\frac{|b-a|}{|a|},$$

where b is the bound under consideration and a is the corresponding actual value.

In PS, the first bound (19) was the best ($\mu = 0.0688$, $\sigma = 0.0436$). Also the simple second bound (19) managed well ($\mu = 0.1814$, $\sigma = 0.0516$). The bound (24) was the second best ($\mu = 0.1007$, $\sigma = 0.0531$), and (25) was the third ($\mu = 0.1078$, $\sigma = 0.0369$).

The simple WS bound (57) was the best in RS ($\mu = 0.2373$, $\sigma = 0.0340$). Our bound (40) was the second ($\mu = 0.3236$, $\sigma = 0.0921$), Parker's bound (42) the third ($\mu = 0.3270$, $\sigma = 0.1227$), and our bound (35) the fourth ($\mu = 0.3385$, $\sigma = 0.2805$).

The bound (57) was the best also in CH ($\mu = 0.2300$, $\sigma = 0.0468$). Our first bound (31) was the second ($\mu = 0.3187$, $\sigma = 0.2854$) and (40) the third ($\mu = 0.3374$, $\sigma = 0.0960$).

The magnitude of many bounds was roughly $\mu \approx 0.5$. An example of a very poor result is the first bound (20) in PS ($\mu = 12.948$, $\sigma = 43.794$). The explanation of this catastrophe is that this bound is always positive, while the actual values were mostly negative.

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