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Common fixed point theorems in cone metric space by altering distances

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Abstract

In this paper we establish some common fixed point theorems in a normal cone metric space by altering distances, also we generalize the MS-altering function (see[4]).

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1 Introduction

Since the Banach Contraction Principle, several types of generalization contraction of Mappings on metric spaces have appeared. One such method of generalization is altering the distances. Delbosco [1] and Skof [6] have established fixed point theorems for self maps of complete metric spaces by altering the distances between the points with the use of a positive real valued function. Huang and Zhang [2] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtained some fixed point results with the assumption that the cone is normal. Recently in [4] author defined a vector valued function, the MS-altering function and proved some fixed point theorems.

In this paper we are generalizing the results of [4] and proving some common fixed point theorems in normal cone metric spaces, also we improve the definition of MS-altering function.

2 Preliminary Notes

Definition 2.1 *: Let E be a real Banach space and P be a subset of E. P is called a cone if*
(a) P is closed, nonempty and $P \neq \{0\}$;
(b) $a, b \in \mathbb{R}, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$;
(c) $x \in P$ and $-x \in P \Rightarrow x = 0$.
Given a cone $P \subseteq E$, we define a partial ordering " $\leq$ " in E by $x \leq y$ if $y - x \in P$. We write $x < y$ to denote $x \leq y$ but $x \neq y$ and $x \ll y$ to denote $y - x \in P^0$, where P^0 stands for the interior of P. We assume cone is solid i.e. that $P^0 \neq \phi$.

Proposition 2.2 *[3]: Let P be a cone in a real Banach space E.*
(1) If $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then $a = 0$.
(2) If $a \in P$ and $a \ll c$, for all $c \in P^0$ then $a = 0$.

A cone P is called normal if there is constant $K > 0$ such that, for all $x, y \in E$, $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. The least value of constant K satisfying this inequality is called the normal constant of P.

Definition 2.3 *[2]: Let X be a nonempty set and E be a real Banach space. Suppose that the mapping $d : X \times X \to E$ satisfies*
(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$, if and only if $x = y$;
(b) $d(x, y) = d(y, x)$, for all $x, y \in X$;
(c) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

For examples of cone metric spaces we refer [2, 5].
Henceforth unless otherwise indicated, P is a normal cone in real Banach space E and " $\leq$ " is partial ordering with respect to P.

Definition 2.4 *[2]: Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.*
(a) If for every $c \in E$ with $0 \ll c$ (or equivalently $c \in P^0$) there is positive integer n_0 such that for all $n > n_0$, $d(x_n, x) \ll c$ then the sequence $\{x_n\}$ converges to x. We denote this by $x_n \to x$, as $n \to \infty$ or $\lim_{n\to\infty} x_n = x$.

(b) If for every $c \in E$ with $0 \ll c$ there is positive integer n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$ then the sequence $\{x_n\}$ is called a Cauchy sequence in X.

(X, d) is called a complete cone metric space, if every Cauchy sequence in X is convergent in X.

Lemma 2.5 *[2]: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$ $(n \to \infty)$.*

Lemma 2.6 *[2]: Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ $(n, m \to \infty)$.*

Definition 2.7 *: A function $f : P \to P$ is called subadditive if for all $x, y \in P$, $f(x + y) \leq f(x) + f(y)$.*

Definition 2.8 *: If Y be any partially ordered set with relation " $\leq$ " and $\psi : Y \to Y$, we say that ψ is non decreasing if $x, y \in Y, x \leq y \Rightarrow \psi(x) \leq \psi(y)$.*

Definition 2.9 *[4]: Let $\psi : P \to P$ be a vector valued function then ψ is called MS-Altering function if*
(a) ψ is non decreasing, subadditive, continuous and sequentially convergent;
(b) $\psi(a) = 0$ if and only if $a = 0$.

We replace conditions (a) and (b) by weaker conditions and define cone altering function as follows

Definition 2.10 *: Let $\psi : P \to P$ be a vector valued function then ψ is called cone altering function if*
(a) ψ is non decreasing, subadditive;
(b) $\psi(a_n) \to 0$ if and only if $a_n \to 0$, for any sequence $\{a_n\}$ in P.

Note that for cone altering function ψ on normal cone P, $\psi(a) = 0$ if and only if $a = 0$.

Definition 2.11 *: Let X be any nonempty set, $f, g : X \to X$ be mappings. A point $w \in X$ is called point of coincidence of f and g if there is $x \in X$ such that $fx = gx = w$.*

Definition 2.12 *: Let X be any nonempty set, $f, g : X \to X$ be mappings. Pair (f, g) is called weakly compatible if $x \in X, fx = gx \Rightarrow fgx = gfx$.*

Lemma 2.13 *: Let (X, d) be a cone metric space and P be a normal cone in a real Banach space E, ψ is is a cone altering function and $k_1, k_2, k > 0$. If $x_n \to x, y_n \to y$ in X and $ka \leq k_1\psi[d(x_n, x)] + k_2\psi[d(y_n, y)]$, then $a = 0$.*

Lemma 2.14 *: Let (X, d) be a cone metric space with normal cone P, and $f, g : X \to X$ be mappings such that, for all $x, y \in X$*

$$\psi[d(fx, fy)] \leq a_1\psi[d(gx, gy)] + a_2\psi[d(fx, gx)] + a_3\psi[d(fy, gy)] + a_4\psi[d(fx, gy)] + a_5\psi[d(fy, gx)] \quad ...(1)$$

where $a_i, i = 1, 2, 3, 4, 5$ are nonnegative constants such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$ and ψ is a cone altering function. If f and g have a point of coincidence then it is unique.

Proof: *Let u is point of coincidence i.e. $v \in X, u = fv = gv$. If u' is another point of coincidence then there is $v' \in X$ such that $u' = fv' = gv'$. Now (1) gives*

$$\begin{aligned}
\psi[d(fv, fv')] &\leq a_1\psi[d(gv, gv')] + a_2\psi[d(fv, gv)] + a_3\psi[d(fv', gv')] \\
&\quad + a_4\psi[d(fv, gv')] + a_5\psi[d(fv', gv)] \\
\psi[d(u, u')] &\leq a_1\psi[d(u, u')] + a_2\psi[d(u, u)] + a_3\psi[d(u', u')] \\
&\quad + a_4\psi[d(u, u')] + a_5\psi[d(u', u)] \\
&= a_1\psi[d(u, u')] + a_4\psi[d(u, u')] + a_5\psi[d(u', u)] \\
&= (a_1 + a_4 + a_5)\psi[d(u', u)]
\end{aligned}$$

since $a_1 + a_4 + a_5 < 1$ hence by proposition 2.2, we have $\psi[d(u', u)] = 0$ i.e. $d(u', u) = 0$ or $u = u'$. Hence point of coincidence is unique.

Proposition 2.15 *: Let X be any nonempty set and $f, g : X \to X$ be mappings. If (f, g) is weakly compatible pair and have a unique point of coincidence then it is unique common fixed point of f and g.*

3 Main Results

Theorem 3.1 *Let (X, d) be a cone metric space with normal cone P, and $f, g : X \to X$ be mappings, $\psi : P \to P$ is cone altering function such that, $f(X) \subset g(X)$, for all $x, y \in X$, (1) is satisfied and $f(X)$ or $g(X)$ is complete, then f and g have a unique point of coincidence. Furthermore if (f, g) is weakly compatible pair then f, g have a unique common fixed point.*

Proof: *Let $x_0 \in X$ be arbitrary, we define sequence y_n such that $y_n = fx_n = gx_{n+1}$ for all $n \geq 0$.*

If $y_n = y_{n+1}$ for any n, then $y_n = y_m$ for all $m > n$ hence $\{y_n\}$ is cauchy sequence.

If $y_n \neq y_{n+1}$ for all n, then from (1)

$$\begin{aligned}
\psi[d(fx_{n+1}, fx_n)] &\leq a_1\psi[d(gx_{n+1}, gx_n)] + a_2\psi[d(fx_{n+1}, gx_{n+1})] + a_3\psi[d(fx_n, gx_n)] \\
&\quad + a_4\psi[d(fx_{n+1}, gx_n)] + a_5\psi[d(fx_n, gx_{n+1})] \\
\psi[d(y_{n+1}, y_n)] &\leq a_1\psi[d(y_n, y_{n-1})] + a_2\psi[d(y_{n+1}, y_n)] + a_3\psi[d(y_n, y_{n-1})] \\
&\quad + a_4\psi[d(y_{n+1}, y_{n-1})]
\end{aligned}$$

writing $d_n = d(y_n, y_{n+1})$ *we have*

$$\begin{aligned}\psi[d_n] &\leq a_1\psi[d_{n-1}] + a_2\psi[d_n] + a_3\psi[d_{n-1}] + a_4\psi[d_n] + a_4\psi[d_{n-1}] \\ (1-a_2-a_4)\psi[d_n] &\leq (a_1+a_3+a_4)\psi[d_{n-1}] \qquad ...(2)\end{aligned}$$

using symmetry of (1) in x, y *we have*

$$(1-a_3-a_5)\psi[d_n] \leq (a_1+a_2+a_5)\psi[d_{n-1}] \qquad ...(3)$$

combining (2) and (3)

$$\psi[d_n] \leq \frac{2a_1+a_2+a_3+a_4+a_5}{2-(a_2+a_3+a_4+a_5)}\psi[d_{n-1}] = \lambda\psi[d_{n-1}]$$

and so $\psi[d_n] \leq \lambda^n\psi[d_0]$, *where* $\lambda = \frac{2a_1+a_2+a_3+a_4+a_5}{2-(a_2+a_3+a_4+a_5)} < 1$.
If $m > n$, *we have*

$$\begin{aligned}\psi[d(y_n, y_m)] &\leq \psi[d(y_n, y_{n+1})] + \psi[d(y_{n+1}, y_{n+2})] + ... + \psi[d(y_{m-1}, y_m)] \\ &\leq \psi[d_n] + \psi[d_{n+1}] + ... + \psi[d_{m-1}] \\ &\leq \lambda^n\psi[d_0] + \lambda^{n+1}\psi[d_0] + ... + \lambda^{m-1}\psi[d_0] \\ &\leq \frac{\lambda^n}{1-\lambda}\psi[d_0]\end{aligned}$$

since $0 \leq \lambda < 1$ *hence by normality of cone* $\| \psi[d(y_n, y_m)] \| \leq \frac{K\lambda^n}{1-\lambda} \| \psi[d_0] \| \to 0$ *therefore* $\psi[d(y_n, y_m)] \to 0$ *and so* $d(y_n, y_m) \to 0$ *hence* $\{y_n\}$ *is a cauchy sequence. Let* $f(X)$ *is complete then since* $y_n = fx_n = gx_{n+1}$ *and* y_n *is cauchy in* $f(X)$, *so it must be convergent in* $f(X)$. *Let* $y_n \to u \in f(X)$*(note that it is also true if* $g(X)$ *is complete with* $u \in g(X)$*). Since* $u \in f(X) \subset g(X)$, *let* $u = g(v)$ *for some* $v \in X$.
we show that $gv = fv$. *Now by (1)*

$$\begin{aligned}\psi[d(fv, u)] &\leq \psi[d(fv, fx_n)] + \psi[d(fx_n, u)] \\ &\leq a_1\psi[d(gv, gx_n)] + a_2\psi[d(fv, gv)] + a_3\psi[d(fx_n, gx_n)] \\ &\quad +a_4\psi[d(fv, gx_n)] + a_5\psi[d(fx_n, gv)] + \psi[d(fx_n, u)] \\ &= a_1\psi[d(u, y_{n-1})] + a_2\psi[d(fv, u)] + a_3\psi[d(y_n, y_{n-1})] \\ &\quad +a_4\psi[d(fv, y_{n-1})] + a_5\psi[d(y_n, u)] + \psi[d(y_n, u)] \\ (1-a_2-a_4)\psi[d(fv, u)] &\leq (a_1+a_3+a_4)\psi[d(u, y_{n-1})] + (a_3+a_5+1)\psi[d(y_n, u)]\end{aligned}$$

hence by lemma 2.13, $\psi[d(fv, u)] = 0$ *and so* $d(fv, u) = 0$ *i.e.* $fv = u = gv$. *Thus* u *is point of coincidence of* f *and* g, *hence by lemma 2.14 it is unique. Furthermore if pair* (f, g) *is weakly compatible then by proposition 2.15,* u *is unique common fixed point of* f *and* g.

If we choose $g = I_X$ i.e. identity mapping of X and $f = T$, we get the main result of [4] as the following corollary.

Corollary 3.2 *[4] Let (X, d) be any complete cone metric space with normal cone P and T be a self map on X, $\psi : P \to P$ be MS-Altering function satisfying*

$$\psi[d(Tx,Ty)] \leq a_1\psi[d(x,y)] + a_2\psi[d(Tx,x)] + a_3\psi[d(Ty,y)] + a_4\psi[d(Tx,y)] + a_5\psi[d(Ty,x)]$$

where $a_i, i = 1, 2, 3, 4, 5$ are nonnegative constants such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then T has a unique fixed point.

If we choose $\psi = I_P$ i.e. identity mapping of P, we get the following corollary.

Corollary 3.3 *Let (X, d) be any cone metric space with normal cone P and f, g be self maps on X such that, $f(X) \subset g(X)$, $f(X)$ or $g(X)$ is complete and,*

$$d(fx, fy) \leq a_1d(gx, gy) + a_2d(fx, gx) + a_3d(fy, gy) + a_4d(fx, gy) + a_5d(fy, gx)$$

where $a_i, i = 1, 2, 3, 4, 5$ are nonnegative constants such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. If pair (f, g) is weakly compatible then f, g have a unique common fixed point.

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