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T- Reich Mapping in Topological Vector Space-Valued Cone Metric Spaces

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Abstract

The object of this paper is to establish some new fixed point results in topological vector space-valued cone metric spaces, by proving the fixed point theorems for T-Reich and T-Kannan contraction mappings in topological vector space-valued cone metric spaces.

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1 Introduction

Huang and Zhang [5] generalized the notion of metric spaces replacing the set of real numbers by an ordered Banach space. Many authors proved fixed point theorems in cone metric spaces (see, e.g. [5, 6, 12]) under additional assumption about the underlying cone, such as normality or even regularity. Recently, Rezapour and Hamlbarani [11] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in cone metric space. In papers [1, 8] authors tried to generalize this approach by using cones in topological vector spaces (tvs) instead of Banach space. However, it should be noted that an old result [9] shows that if the underlying cone of an ordered tvs is solid and normal it must be an ordered normed space. So proper generalizations when passing from norm-valued cone metric space can be obtained only in the case of non normal cones. Recently Kadelburg et. al. [8] developed further theory of tvs-cone metric space and proved some fixed point results and common fixed point results in tvs-cone metric space. In this paper we prove some fixed point theorem for T-Reich type mappings and T-Kannan type contraction [6] in tvs-valued cone metric space.

2 Preliminary Notes

Definition 2.1. *Let E be a real Hausdorff topological vector space (tvs for short) with the zero vector θ . A nonempty proper and closed subset P of E is called a (convex) cone if $P + P \subset P, \lambda P \subset P$ for $\lambda \geq 0$ and $P \cap \{-P\} = \{\theta\}$. We will always assume that P^0 is non empty (here P^0 denotes the interior of P), and such cones are called solid.*

Each cone P includes a partial order " $\leq$ " on E defined by $x \leq y \Leftrightarrow y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$. The pair (E, P) is an ordered topological vector space.

Let P be a cone in a real Banach space E then P is called normal, if there exists a constant $K > 0$ such that for all $x, y \in E$ and $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least positive number K satisfying the above inequality is called the normal constant of P.

Proposition 2.2. *[8] Let P be a cone in a real tvs E. If for $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then $a = \theta$.*

Definition 2.3. *Let X be a nonempty set and (E, P) an ordered tvs. A function $d : X \times X \to E$ is called tvs-cone metric and (X, d) is called tvs-cone metric space, if the following conditions hold:*
(a) $\theta \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = \theta \Leftrightarrow x = y$,
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let $x \in X$ and $\{x_n\}$ be a sequence in X. Then it is said the following:
(a) $\{x_n\}$ *tvs*-cone converges to x, if for every $c \in P^0$, there is a natural number n_0 such that $d(x_n, x) \ll c$, for all $n > n_0$. We denote it by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(b) $\{x_n\}$ is a *tvs*-cone Cauchy sequence if for every $c \in P^0$ there is a natural

number n_0 such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.
(c) (X, d) is *tvs*-cone complete if every *tvs*-cone Cauchy sequence is *tvs*-cone convergent in X.

In further discussion we always assume that E is a real *tvs* and P is a cone in E, and " $\leq$ " is partial ordering with respect to P.

Lemma 2.4. *[8] (a) Let $\theta \leq x_n \to \theta$ in (E, P) and $\theta \ll c$. Then there is n_0 such that $x_n \ll c$ for every $n > n_0$.*
(b) It can happened that $\theta \leq x_n \ll c$ for each $n > n_0$, but $x_n \nrightarrow \theta$ in (E, P).
(c) It can happened that $x_n \to x, y_n \to y$ in the tvs-cone metric d, but that $d(x_n, y_n) \nrightarrow d(x, y)$ in (E, P).
(d) $\theta \leq u \ll c$ for each $c \in P^0 \Rightarrow u = \theta$.
(e) $x_n \to x, x_n \to y$ (in the tvs-cone metric)$\Rightarrow x = y$.

Lemma 2.5. *[8] (a) If $u \leq v$ and $v \ll w$, then $u \ll w$,*
(b) If $u \ll v$ and $v \leq w$, then $u \ll w$,
(c) If $u \ll v$ and $v \ll w$, then $u \ll w$,
(d) Let $x \in X$, and $\{x_n\}$ and $\{b_n\}$ be two sequences in X and E respectively, $\theta \ll c$ and $\theta \leq d(x_n, x) \leq b_n$ for all n. If $b_n \to \theta$, then there is n_0 such that $d(x_n, x) \ll c$ for all $n > n_0$.

Definition 2.6. *Let T and f are two self maps of a tvs-valued cone metric space X. Then (T, f) is called a Banach pair, if $fTx = Tfx$ for every $x \in F(f)$, where $F(f)$ is the set of all fixed point of f.*

In further discussion we write "0" in place of zero vector "θ" of E.

Definition 2.7. *Let (X, d) be a tvs-cone metric space and $T, f : X \to X$ satisfy, $d(Tfx, Tfy) \leq ad(Tx, Ty) + bd(Tx, Tfx) + cd(Ty, Tfy)$ for all $x, y \in X$, where a, b, c are nonnegative constants such that $a+b+c < 1$. Then f is called T-Reich mapping.*

3 Main Results

Theorem 3.1. *Let (X, d) be a complete tvs-cone metric space and $T, f : X \to X$ and f is T-Reich mapping, T, f are continuous, T is injective and sub-sequentially convergent mapping, then f has a fixed point in X. Moreover, if (T, f) is a Banach pair then T and f have a unique common fixed point in X.*

Proof: *Let $x_0 \in X$ be arbitrary, we define a sequence $\{x_n\}$ by $x_{n+1} = fx_n$, for all $n \geq 0$. Now since f is T-Reich mapping hence we have,*

$$\begin{aligned} d(Tfx_n, Tfx_{n-1}) &\leq ad(Tx_n, Tx_{n-1}) + bd(Tx_n, Tfx_n) + cd(Tx_{n-1}, Tfx_{n-1}) \\ d(Tx_{n+1}, Tx_n) &\leq ad(Tx_n, Tx_{n-1}) + bd(Tx_n, Tx_{n+1}) + cd(Tx_{n-1}, Tx_n) \end{aligned}$$

Writing $d_n = d(Tx_{n+1}, Tx_n)$ *we have,*

$$\begin{aligned} d_n &\leq ad_{n-1} + bd_n + cd_{n-1} \\ (1-b)d_n &\leq (a+c)d_{n-1} \\ d_n &\leq \frac{a+c}{1-b}d_{n-1} \\ d_n &\leq \lambda d_{n-1} \end{aligned}$$

where $\lambda = \frac{a+c}{1-b} < \frac{a+c}{a+c} < 1$. *Hence* $\lambda < 1$ *and* $d_n \leq \lambda^n d_0$, *where* $d_0 = d(x_1, x_0)$. *Now if* $m, n \in \mathbb{N}$ *and* $m > n$ *then we have,*

$$\begin{aligned} d(Tx_n, Tx_m) &\leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m) \\ d(Tx_n, Tx_m) &\leq d_n + d_{n+1} + d_{n+2} + \cdots + d_{m-1} \\ d(Tx_n, Tx_m) &\leq \lambda^n d_0 + \lambda^{n+1} d_0 + \lambda^{n+2} d_0 + \cdots \\ d(Tx_n, Tx_m) &\leq \lambda^n d_0 [1 + \lambda + \lambda^2 + \cdots] \\ d(Tx_n, Tx_m) &\leq \frac{\lambda^n d_0}{1-\lambda} \to 0 \quad as \quad n \to \infty \quad (since \lambda < 1). \end{aligned}$$

Now using properties (a) of Lemma 2.4, and only the assumption that the underlying cone is solid, we have, for every $e \in P^0$ *there is* n_0 *such that* $\frac{\lambda^n d_0}{1-\lambda} \ll e$ *for all* $n > n_0$ *and by (a) of Lemma 2.5, we conclude that* $\{Tx_n\}$ *is a Cauchy sequence. Since* X *is complete we must have* $u \in X$, *such that* $Tx_n \to u$ *as* $n \to \infty$.
Now since T *is sub-sequentially convergent, therefore the sequence* $\{x_n\}$ *has a convergent subsequence* $\{x_{n_k}\}$ *such that* $x_{n_k} \to z \in X$, *also* T *is continuous hence* $Tx_{n_k} \to Tz$ *as* $k \to \infty$, *and by the uniqueness of limit in tvs-cone metric space we have* $Tz = u$.
Now since f *is continuous and* $x_{n_k} \to z$ *so* $fx_{n_k} \to fz$ *and by continuity of* T *we have* $Tfx_{n_k} \to Tfz$.

Now we show that $Tfz = fz$. *Then we have*

$$\begin{aligned} d(Tfz, Tz) &\leq d(Tfz, Tfx_{n_k}) + d(Tfx_{n_k}, Tz) \\ &\leq ad(Tz, Tx_{n_k}) + bd(Tz, Tfz) + cd(Tx_{n_k}, Tfx_{n_k}) \\ &\quad + d(Tfx_{n_k}, Tz) \\ &= ad(Tz, Tx_{n_k}) + bd(Tz, Tfz) + cd(Tx_{n_k}, Tx_{n_k+1}) \\ &\quad + d(Tx_{n_k+1}, Tz) \\ (1-b)d(Tz, Tfz) &\leq ad(Tz, Tx_{n_k}) + cd_{n_k} + d(Tx_{n_k+1}, Tz) \\ d(Tz, Tfz) &\leq \frac{a}{1-b}d(Tz, Tx_{n_k}) + \frac{c}{1-b}d_{n_k} + \frac{1}{1-b}d(Tx_{n_k+1}, Tz) \end{aligned}$$

Now since $Tx_{n_k} \to Tz$ *and* $d_{n_k} \to 0$ *as* $k \to \infty$, *hence for any given* $e \in P^0$ *we can choose* n_1 *such that* $d(Tz, Tx_{n_k}) \ll \frac{1-b}{3a}e$, $d_{n_k} \ll \frac{1-b}{3c}e$ *and* $d(Tfx_{n_k+1}, Tz) \ll$

$\frac{1-b}{3}e$ for all $k > n_1$. Hence we have
$d(Tz, Tfz) \ll \frac{e}{3} + \frac{e}{3} + \frac{e}{3} = e$ for all $e \in P^0$.
So we have $d(Tz, Tfz) = 0$. Hence $Tfz == Tz$, but T is injective hence $fz = z$ i.e. z is a fixed point of f.
Now we show that z is unique fixed point of f. Let w is another fixed point of f then we have $fw = w$ and

$$\begin{aligned} d(Tz, Tw) &= d(Tfz, Tfw) \\ &\leq ad(Tz, Tw) + bd(Tz, Tfz) + cd(Tw, Tfw) \\ &= ad(Tz, Tw) + bd(Tz, Tz) + cd(Tw, Tw) \\ &= ad(Tz, Tw) \end{aligned}$$

since $0 \leq a < 1$, hence by proposition 2.2, we must have $d(Tz, Tw) = 0$ i.e. $Tz = Tw$, and T is injective hence $z = w$. Thus fixed point is unique.

Now if (T, f) is a Banach pair, then T and f commutes at the fixed point of f, which implies that $fTz = Tfz$ i.e. $fTz = Tz$. It shows that Tz is another fixed point of f. Hence by uniqueness of fixed point of f we must have $Tz = z$ i.e. z is also a fixed point of T, and by uniqueness of fixed point of f, it is unique common fixed point of f and T. □

The following corollary extends the main result of Beiranvand [2] to the *tvs*-cone metric space.

Corollary 3.2. *(T-contraction) Let (X, d) be a complete tvs-cone metric space and $T, f : X \to X$ satisfy, $d(Tfx, Tfy) \leq ad(Tx, Ty)$, for all $x, y \in X$ where $0 \leq a < 1$. If the mapping T and f are continuous and T is injective, sub-sequentially convergent mapping then f has a fixed point in X. Moreover, if (T, f) is a Banach pair then T and f have a unique common fixed point in X.*
Proof: *The proof of corollary follows by taking $b = c = 0$, in theorem 3.1.*

Corollary 3.3. *(Reich type) Let (X, d) be a complete tvs-cone metric space and $f : X \to X$ satisfies $d(fx, fy) \leq d(x, y) + bd(x, fx) + cd(y, fy)$, for all $x, y \in X$, where $a, b, c \geq 0$ with $a + b + c < 1$. If the mapping f is continuous then f has a unique fixed point in X.*
Proof: *The proof of this corollary follows by taking $T = I_X$ in theorem 3.1.*

Corollary 3.4. *(T-Kannan type) Let (X, d) be a complete tvs-cone metric space and $T, f : X \to X$ satisfy $d(Tfx, Tfy) \leq b[d(Tx, Tfx) + d(Ty, Tfy)]$, for all $x, y \in X$, where $b \in [0, \frac{1}{2})$. If the mappings T and f are continuous and T is injective, sub-sequentially convergent mapping then f has a unique fixed point in X. Moreover if (T, f) is a Banach pair then T and f have a unique common fixed point in X.*

Proof: *The proof of this corollary follows by taking* $b = c, a = 0$ *in theorem 3.1.*

Example 3.5. *Let* $E = (C_{[0,1]}, \mathbb{R}), P = \{\varphi \in E : \varphi \geq 0\}$, $X = [0,1]$ *and* $d : X \times X \to E$ *is defined by* $d(x,y)(t) = |x-y|e^t$ *where* $e^t \in E$. *Define* $T, f : X \to X$ *such that* $fx = \frac{x}{2}, Tx = \frac{x}{3}$, *then*
$d(Tfx, Tfy) = d(\frac{x}{6}, \frac{y}{6}) = \frac{1}{6}|x-y|e^t \leq \frac{1}{3}|x-y|e^t = d(Tx, Ty)$
Let $a = \frac{1}{2}, b = \frac{1}{6}, c = \frac{1}{5}$, *then clearly* T *is injective, sub-sequentially convergent and* (T, f) *is a Banach pair, hence all the conditions of theorem 3.1 are satisfied, and* $x = 0$ *is the required unique common fixed point.*

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