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Structural Properties of Polynomial and Rational Matrices, a survey

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Abstract

A review of the structural properties of polynomial and rational matrices is presented. After the analysis of the finite spectrum of a polynomial matrix $A(\lambda)$, via the Smith canonical form, we analyze the infinity as an eigenvalue, but also as a pole or zero (via the Smith McMillan canonical form) when considering $A(\lambda)$ in the set of rational matrices. Then we focus on the structures generated by the columns of $A(\lambda)$. Here we review two different approaches: when considering linear combinations over the rational functions, and when linear combinations are supposed to be over polynomials only. The objective is to compare and contrast the results of these two lines of thought, as well as to underline the fundamental differences between matrix polynomials in one or several variables. Structure preserving transformations and equivalence relations over polynomial matrices are also reviewed. With this objective in mind, we also give some insights on the eigenvalue structure of multivariable matrix polynomials.

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1 Introduction

1.1 Motivation

Polynomials and polynomial matrices arise naturally as modeling tools in several areas of applied mathematics, sciences and engineering, specially in systems theory [8, 26, 27]. It is critical to understand their structural properties as well as their physical meaning depending on the context. A few examples are as follows: the pole-zero structure of a linear system, represented by the state space matrices (A, B, C, D) or the transfer matrix function $H(\lambda) = C(\lambda I - A)^{-1}B + D$ (a rational matrix in the indeterminate λ), can be recovered from the eigenstructure of some polynomial matrices [10]. This structural information is important in problems such as decoupling [28]. In particular, the degrees of the vectors in a minimal polynomial basis of the null-space of the pencil

$$P(\lambda) = \begin{bmatrix} \lambda I - A & B \\ C & D \end{bmatrix},$$

which are defined as the Kronecker invariant indices [24], correspond to the invariant lists defined in [18]. These invariants are key information when solving problems of structural modification for linear dynamic systems [15].

Computing polynomial null-space basis is also important when solving the problem of column reduction of a polynomial matrix [19]. Column reduction is the initial step in several elaborated algorithms in computer-aided control system design.

With the polynomial equation approach of control theory, introduced by Kučera [12], the solution of several control problems has been reformulated in terms of polynomial matrix equations or Diophantine equations [13]. Such formulations are also relevant in the behavioral approach of J. C. Willems and co-workers [20]. The solutions of these polynomial matrix equations are based on important structural properties of the involved matrices. *A review of the structural properties of polynomial matrices, as well as structure preserving transformations of polynomial matrices, will be one of the two main topics of this work.*

At the present, the amount of literature concerning polynomial matrices and their applications is really wide. A lot of work has been done since the last century, some classical references are for example [6, 16, 17, 22]. Most of the work done focus on the mathematical nature of the polynomial matrix (or the rational matrix in general) as well as on its properties as a matrix function acting over linear spaces. In the case the polynomial matrix is considered as a function acting on $\mathbb{R}_n$ or $\mathbb{C}_n$ (the n dimensional vectors with reals or complex entries), the analysis becomes an extension of the classical spectral theory of real or complex matrices [7, 8]. Another natural point of view is that of transformations of vectors of polynomial or rational functions, say $\mathcal{P}_n$ or $\mathcal{R}_n$, respectively.

The essential algebraic distinction to be made concerns, then, operators acting on a linear space over $\mathbb{R}$, $\mathbb{C}$, or $\mathcal{R}$ (with a concomitant structure of subspaces), or on a module over $\mathcal{P}$ (with a concomitant structure of submodules). *It is our sense that the literature has bifurcated in these two directions and that there is a case to be made for a review comparing and contrasting results of these two lines of thought. The other main topic of our work concerns this review.*

1.2 Notations and first discussions

We are interested in different classes of matrix valued functions, and it could be helpful to encapsulate their properties in the form of basic structures of abstract algebra. Although we do not delve deeply into these structures, we suggest [9] and the classic texts [2, 4] for definitions and basic properties.

Consider $m \times n$ matrices, A, whose entries are either real or complex valued functions of a scalar variable λ (to be understood from the context). Write $A(\lambda)$ for such a matrix. If the elements of A are polynomials in λ, then $A(\lambda)$ is said to be a polynomial matrix. The set of all such $m \times n$ polynomial matrices (with m and n fixed) forms a group under matrix addition and is denoted by $\mathcal{P}_{m\times n}$. When scalar multiplication is introduced (over the real numbers, $\mathbb{R}$, or the complex numbers, $\mathbb{C}$, as appropriate), then $\mathcal{P}_{m\times n}$ becomes a linear space. When $m = n$ then, together with matrix multiplication, $\mathcal{P}_{n\times n}$ has the structure of a ring, with the obvious identity I. A polynomial matrix can also be seen as a polynomial with matrix coefficients, i.e. a matrix polynomial. In this survey we use both terms indistinctively.

More generally, we consider $m \times n$ matrices whose entries are scalar *rational* functions in their lowest terms. The additive group of all such matrices is denoted by $\mathcal{R}_{m\times n}$. Clearly, $\mathcal{P}_{m\times n} \subset \mathcal{R}_{m\times n}$. Once again, if $m = n$, $\mathcal{R}_{n\times n}$ has the structure of a ring with identity I. Furthermore, if $m = n = 1$, then $\mathcal{R}_{1\times 1}$ (the scalar rational functions) form a field (all nonzero members have a multiplicative inverse). Of course, the same is not true of $\mathcal{P}_{1\times 1}$ (the scalar polynomials), which is a ring. In what follows we will use $\mathcal{P}$ and $\mathcal{R}$ to refer to the sets of scalar polynomials and rational functions, respectively. Similarly, $\mathcal{P}_{n\times 1}$ or $\mathcal{R}_{n\times 1}$ will be written as $\mathcal{P}_n$ or $\mathcal{R}_n$. Furthermore, in the later sections of this paper, $\mathcal{P}^t$ and $\mathcal{P}^t_{m\times n}$ will be used to refer to the sets of scalar and matrix polynomials in t variables, respectively.

Some of the ideas in this survey are common to different subject areas, and so, have received different treatments or emphasis. Valuable references which deal with our topic, but from quite different points of view are [7, 10, 16, 21]. Here we focus on properties of matrix polynomials that are invariant under equivalence transformations. These structural elements include the set of eigenvalues and their multiplicities (including the eigenvalue at infinity), and the structures related to the columns (rows) of the matrix when considering linear combinations over $\mathcal{P}$ or over $\mathcal{R}$.

Here we want to underline that $\mathcal{R}_{m\times n}$ is a linear space over the field $\mathcal{R}$ (and also over $\mathbb{R}$, or over $\mathbb{C}$, depending on the context). In contrast, as $\mathcal{P}$ is not a field, $\mathcal{P}_{m\times n}$ is a module over the ring $\mathcal{P}$. This fact will be important when describing structures related to the columns (rows) of the polynomial matrix $A(\lambda)$. For example, we will define the *nullspace* of $A(\lambda)$ when $A(\lambda)$ is considered as an element of $\mathcal{R}_{m\times n}$, or the *syzygy module* of $A(\lambda)$ when $A(\lambda) \in \mathcal{P}_{m\times n}$. Thus, we bring together two different approaches to the analysis of matrix polynomials: when considered as members of the linear space $\mathcal{R}_{m\times n}$, or as members of the module $\mathcal{P}_{m\times n}$. We also emphasize the importance of the module context when analyzing matrix polynomials in several variables.

1.3 Outline

Our survey is organized in five main sections. Section 2 provides an introduction to the ideas to be further developed. Section 3 is about canonical forms revealing the eigenvalue structure of a polynomial matrix $A(\lambda)$. In Section 4 we present other canonical forms providing more information on the invariant subspaces generated by $A(\lambda) \in \mathcal{R}_{m\times n}$, and we relate these forms to the idea of Gröbner bases and sub-modules. Some equivalence relations and eigenvalue structure preserving transformations are presented in Section 5. Final remarks and conclusions are given in Section 6.

Throughout the different sections, the sub-sections are related as follows:

§2.1 provides a quick introduction to matrix polynomials and their eigenvalues (for the regular case) including the eigenvalue at infinity. The extension to the finite eigenvalues in non-regular cases is carried out in §3.1 in terms of the Smith canonical form. In §3.2 we extend the concept of the eigenvalue at infinity to the non-regular cases together with an alternative way to analyze the point at infinity when considering $A(\lambda) \in \mathcal{R}_{m\times n}$. This includes the Smith-McMillan form, and the Smith-McMillan canonical form at infinity.

§2.2 and §2.3 concern rational matrices and their null spaces, modules and syzygy modules. We show that, when considering the structures related to the columns of $A(\lambda)$, we have different properties when linear combinations are taken over $\mathcal{P}$ or $\mathcal{R}$ (especially for the case of multivariable polynomials). In §4.1 we introduce other canonical forms including the Hermite form, which gives a polynomial basis $B(\lambda)$ for the column subspace of $A(\lambda)$. In §4.2 we show that this basis is, in fact, the Gröbner basis of the sub-module generated by the columns of $A(\lambda)$ when considering linear combinations over $\mathcal{P}$.

§5.1 formalizes the equivalence relations defined via the Smith and the Smith-McMillan canonical forms. In §5.2 we generalize these equivalence relations to the case where the equivalent matrices are of different size. Some examples related to matrix polynomial linearizations are also presented in this sub section. Finally, in §5.3 we briefly review some concepts concerning the structure of matrix polynomials in several variables. The objective here is to present other transformations of matrix polynomials which preserve both the finite and the infinite spectral properties.

2 Preliminaries

2.1 Matrix polynomials

The natural idea of *degree* is attached to a matrix polynomial $A(\lambda)$: if we write $A(\lambda)$ in terms of the powers of λ, $A(\lambda) = \sum_{j=0}^{d} A_j \lambda^j$ where $A_d \neq 0$, then d is the degree of $A(\lambda)$. In the linear space $\mathcal{P}_{m\times n}$ over $\mathbb{C}$ there is a useful structure of subspaces obtained by confining attention to all such functions with degree *not exceeding* $d = 0, 1, 2, \ldots$

When $m = n$ and the *characteristic polynomial* $\psi(\lambda) = \det(A(\lambda))$ is not identically zero, $A(\lambda)$ is said to be *regular* or *nonsingular*. The zeros of the characteristic polynomial $\psi(\lambda)$ are known as the *(finite) eigenvalues* of $A(\lambda)$. If α is a finite eigenvalue of $A(\lambda)$, then $\operatorname{rank}(A(\alpha)) < n$ and there exist nonzero vectors u_i for which $A(\alpha)u_i = 0$ called the *(right) eigenvectors* associated with α. The *geometric multiplicity* m_g of a finite eigenvalue α is given by $m_g = n - \operatorname{rank}(A(\alpha))$. The *algebraic multiplicity* m_a is equal to the multiplicity of α as a zero of $\psi(\lambda)$.

When α is a multiple eigenvalue (i.e. a multiple zero of $\psi(\lambda)$), the concept of eigenvector has to be generalized. Here, we do it using a familiar idea from elementary ordinary differential equations. Consider the homogeneous equation

$$A(\frac{d}{dt})x(t) = 0. \tag{1}$$

We look for a solution in the form $x(t) = c(t)e^{\alpha t}$ and, for convenience, we can be more explicit about the form of $c(t)$ and write

$$x(t) = \left(\frac{t^k}{k!}u_0 + \frac{t^{k-1}}{(k-1)!}u_1 + \cdots + u_k\right)e^{\alpha t} \tag{2}$$

where $u_0 \neq 0$. Then it is easy to verify the following proposition (Proposition 1.9 of [7]):

Proposition 2.1 *The function $x(t)$ of (2) is a solution of (1) if and only* *if*

$$\sum_{q=0}^{i} \frac{1}{q!} A^{(q)}(\alpha) u_{i-q} = 0, \quad \text{for} \quad i = 0, 1, \ldots, k. \tag{3}$$

A sequence $u_0 \neq 0, u_1, \ldots, u_k$ satisfying the $k+1$ equations (3) is called a *Jordan chain* of length $k+1$ for $A(\lambda)$, associated to α. The number of such chains and their lengths (known as *partial multiplicities*) determine the algebraic and geometric multiplicities, m_a and m_g, of α (for more details see the following sections). If, on the other hand, $\text{rank}(A(\lambda)) = n$ for all λ, then $\det(A(\lambda))$ is a nonzero constant and matrix $A(\lambda)$ has a polynomial inverse. Polynomial matrices with polynomial inverse are said to be *unimodular matrices*. Unimodular matrices are *units* in the ring of square polynomial matrices, and they have no finite eigenvalues.

With a view to examining the structure of a polynomial $A(\lambda)$ at infinity, consider the matrix polynomial

$$A_*(\lambda) := A_0 \lambda^d + A_1 \lambda^{d-1} + \cdots + A_d = \sum_{i=0}^{d} A_i \lambda^{d-i}, \tag{4}$$

called the *reverse* or *dual* polynomial of $A(\lambda)$. Notice that

$$A_*(\lambda) = \lambda^d A(1/\lambda) = \lambda^d \bar{A}(\lambda) \tag{5}$$

and so, if α is a non-zero eigenvalue of $A_*(\lambda)$ with geometric and algebraic multiplicities m_g and m_a, then $1/\alpha$ is an eigenvalue of $A(\lambda)$ with the same multiplicities.

Clearly, the $n \times n$ polynomial $A(\lambda)$ has a zero eigenvalue if and only if A_0 is singular and, in this case, we say that $A_*(\lambda)$ has an *eigenvalue at infinity* and the multiplicities of this eigenvalue are defined to be just those of the zero eigenvalue of $A(\lambda)$. Similarly, if A_d is singular, $A(\lambda)$ is said to have an eigenvalue at infinity. Since $A_*(0) = A_d$, $A(\lambda)$ has no eigenvalue at infinity if and only if A_d is nonsingular. In particular, *monic* matrix polynomials (when $A_d = I$) have no eigenvalue at infinity.

Example 2.2 *If*

$$U(\lambda) = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \quad \textit{then} \quad U_*(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}. \tag{6}$$

In this case $U(\lambda)$ has degree one and no finite eigenvalue (it is unimodular), but it has an eigenvalue at infinity with $m_g = 1,\ m_a = 2$.

The eigenvalues and their multiplicities, together with the associated eigenvectors and the partial multiplicities, form what we call the *eigenvalue structure* or *eigenstructure* of $A(\lambda)$. Obtaining the full eigenvalue structure of a matrix polynomial $A(\lambda)$ is a delicate task to be reviewed in the following sections. In the process, it will be shown that the notions of eigenvalue and Jordan chain can be extended to the cases of rectangular and singular matrix polynomials.

2.2 Rational matrices

Of course, polinomial matrices are also rational matrices (with denominators identically equal to one) and it is useful to consider them in this context. This is the case when dealing with the concept of rank and nullspace, since the basic algebraic setting is the linear space $\mathcal{R}_{m\times n}$ over the field $\mathcal{R}$. This allows us take advantage of the theory of linear spaces.

The *algebraic rank* of $A(\lambda) \in \mathcal{R}_{m\times n}$ is r if there is at least one $r \times r$ minor of $A(\lambda)$ which is not identically zero and every $q \times q$ minor of $A(\lambda)$ with $q > r$ (if any) is identically equal to zero. In contrast, the *geometric rank* of $A(\lambda)$ is defined as the row (or column) rank of $A(\lambda)$, namely, the number of rows (or columns) of $A(\lambda)$ linearly independent over $\mathcal{R}$. However, it is easy to prove that the geometric rank and the algebraic rank are equal, and so we can refer to the rank r of $A(\lambda)$ without ambiguity. Similarly, $A(\lambda) \in \mathcal{R}_{n\times n}$ is said to be regular or nonsingular if $r = n$ and, in this case, $A(\lambda)^{-1} \in \mathcal{R}_{n\times n}$.

If $A(\lambda) \in \mathcal{R}_{m\times n}$ has rank $r \leq \min(m, n)$, then $A(\lambda)$ could have non-trivial *nullspaces*. The *right* nullspace of $A(\lambda)$ or *kernel* of $A(\lambda)$, $\ker A(\lambda)$, is defined to be the set of all (column) vectors $z(\lambda) \in \mathcal{R}_n$ such that $A(\lambda)z(\lambda) = 0$, i.e.

$$\ker A(\lambda) = \{z(\lambda) \in \mathcal{R}_n \mid A(\lambda)z(\lambda) = 0\}.$$

This subspace can be generated by $n - r$ linearly independent vectors

$$z_1(\lambda), \dots, z_{n-r}(\lambda) \in \mathcal{R}_n,$$

satisfying $A(\lambda)z_i(\lambda) = 0$ for each i. Then the columns of $Z = [z_1(\lambda), \dots, z_{n-r}(\lambda)]$ form a *basis* for $\ker A(\lambda)$, i.e.

$$\ker A(\lambda) = \Big\{ \sum_{i=1}^{n-r} c_i z_i(\lambda) \mid c_i \in \mathcal{R} \Big\} = \operatorname{span}_{\mathcal{R}}\{z_1(\lambda), z_2(\lambda), \dots, z_{n-r}(\lambda)\} = \langle Z \rangle_{\mathcal{R}}.$$

The dimension of $\ker A(\lambda)$, $\nu = \operatorname{rank}(Z)$, is called the nullity of $A(\lambda)$, and the familiar relation $n = \nu + r$ is verified. The *left* nullspace of $A(\lambda)$ is the analogous subspace of $\mathcal{R}_m$ associated with the transpose, $A^T(\lambda)$.

Now notice that, by multiplying by a suitable scalar polynomial, any basis of a subspace $\mathcal{S}$ of $\mathcal{R}_n$ (over $\mathcal{R}$) can be transformed to a strictly polynomial basis, i.e. a basis with all its elements being polynomials. However, the degrees of the polynomials in such a basis are to be controlled. Thus, (as in [5]):

Definition 2.3 *If $A(\lambda) \in \mathcal{R}_{m\times n}$ has rank r, then a* minimal polynomial basis *for ker$A(\lambda)$ consists of a set of polynomials $z_1(\lambda), \ldots, z_{r-n}(\lambda) \in \mathcal{P}_n$ forming a basis for ker$A(\lambda)$ and with the property that, if δ_i is the degree of $z_i(\lambda)$ for each i, then $\sum_{i=1}^{n-r} \delta_i$ is minimal over the choice of all polynomial basis.*

If a polynomial basis is obtained from a general rational basis Z_R by multiplying by a common multiple of all the denominators of Z_R then, as illustrated in Example 2.6 below, the resulting polynomial basis is not necessarily minimal. The following argument provides some insight into the problem, but a more practical construction of a minimal basis appears as Theorem 2.5 below.

Let $A(\lambda) \in \mathcal{R}_{m\times n}$. Given any polynomial basis $Z := \{z_1(\lambda), \ldots, z_{n-r}(\lambda)\}$ for ker$A(\lambda)$, a *minimal* polynomial basis can be constructed as follows:

1. Choose a vector $z_j(\lambda)$ of minimal degree from Z and call it $f_1(\lambda)$.
2. Consider the ser $\{v_j \in \ker A(\lambda) \mid v_j \notin \text{span}_{\mathcal{R}}\{f_1\}\}$ and choose one polynomial vector, say $f_2(\lambda)$, of minimal degree.
3. From $\{v_j \in \ker A(\lambda) \mid v_j \notin \text{span}_{\mathcal{R}}\{f_1, f_2\}\}$ choose one polynomial vector of minimal degree and call it $f_3(\lambda)$.
4. Continue in this way for $n - r$ steps.

A set $f_1, f_2, \ldots, f_{n-r}$ constructed in this way is a minimal polynomial basis for ker$A(\lambda)$. For a formal proof see Section 6.5.4 of [10], for example; and Section 6.3.2 for the following concept.

Definition 2.4 *Let $A(\lambda) \in \mathcal{P}_{m\times n}$ and t_j be the highest power of λ appearing in column j, $1 \le j \le n$. Form an $m \times n$* associated matrix *D_A of integers d_{ij} as follows:*
If $deg(a_{ij}(\lambda)) = t_j$ then d_{ij} is the coefficient of λ^{t_j} in $a_{ij}(\lambda)$.
If $deg(a_{ij}(\lambda)) < t_j$ then $d_{ij} = 0$.
The matrix $A(\lambda) \in \mathcal{P}_{m\times n}$ is column reduced *(or* column proper*) if the associated matrix D_A has rank n.*

As an example, if $A(\lambda) = \begin{bmatrix} 7\lambda^2 - 1 & \lambda^2 \\ \lambda^2 - s + 3 & -3\lambda^3 + \lambda^2 - 1 \end{bmatrix}$, then $D_A = \begin{bmatrix} 7 & 0 \\ 1 & -3 \end{bmatrix}$, and $A(\lambda)$ is clearly column reduced.

With these ideas we can now formulate a practical criterion for a polynomial basis of a subspace of $\mathcal{R}_n$ to be minimal. For more details, and a proof of the following results see §6.4 in [10], for example.

Theorem 2.5 *Let $S = \langle Z \rangle_{\mathcal{R}}$ be a subspace of $\mathcal{R}_n$, and let $Z = [z_1(\lambda), \ldots, z_q(\lambda)]$ be a polynomial basis for S, $(1 \leq q \leq n)$. Then Z is a minimal polynomial basis of S if and only if*

i) rank$(Z(\lambda)) = q$ (over $\mathbb{C}$) for all λ and

ii) $Z(\lambda)$ is column reduced.

In the special case $q = n$ condition i) simply says that $Z(\lambda)$ is unimodular.

Example 2.6 *Consider the matrix*

$$A(\lambda) = \begin{bmatrix} \lambda+1 & \lambda^2 & \lambda^2+s+1 & \lambda^3 \\ 0 & 1 & 1 & \lambda \\ -\lambda & \lambda+1 & 1 & \lambda^2+s \end{bmatrix}.$$

It is easily verified that the rank of $A(\lambda)$ is two and a rational basis of its null-space is given by the columns of

$$Z_R = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ -\lambda^{-1} & \lambda^{-1} \end{bmatrix}.$$

A polynomial basis is determined by the columns of

$$\bar{Z}_P = \lambda(\lambda-1)Z_R = \begin{bmatrix} -\lambda(\lambda-1) & 0 \\ 0 & -\lambda(\lambda-1) \\ \lambda(\lambda-1) & 0 \\ -(\lambda-1) & (\lambda-1) \end{bmatrix},$$

where $\lambda(\lambda-1)$ is a common multiple of the denominators of Z_R. Clearly this basis is not minimal since the sum of the degrees of the columns of $\bar{Z}_p$ exceeds that of

$$Z_P = \lambda Z_R = \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \\ \lambda & 0 \\ -1 & 1 \end{bmatrix},$$

where λ is the least common multiple of the denominators of Z_R. However, since Z_P loses rank at $\lambda = 0$, it also fails to determine a minimal basis. We can verify that a minimal basis for the null space of $A(\lambda)$ is given by the columns of

$$Z = \begin{bmatrix} 1 & 0 \\ 1 & -\lambda \\ -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now, how can we extend the discussion of nullspaces presented above, to the case when $A(\lambda)$ is viewed as an element of $\mathcal{P}_{m\times n}$ (a *module* over the ring $\mathcal{P}$)? We briefly answer this question in the next subsection. It is important to state, however, that this dichotomy is meaningless when talking about the eigenvalues of $A(\lambda)$. Notice that, with the definition of algebraic rank, the eigenvalue structure is determined by the ranks and null spaces of constant matrices whether $A(\lambda)$ is in $\mathcal{P}_{m\times n}$ or in $\mathcal{R}_{m\times n}$. The theory of sub-modules is important when considering the structures generated by the columns (or rows) of a given matrix polynomial. This theory has been used to analyze matrix polynomials from an abstract point of view and has important applications in linear systems theory (among other areas of applied mathematics). Moreover, this theory of modules applies more generally to the case of multivariate polynomials (see [3], [23]).

2.3 Modules

An *R-module* module M is a set of elements which, together with the two operations of addition and scalar multiplication, has the same defining properties as a linear space, *except* that the underlying scalars are taken from a ring R rather than a field. The concept of "module" is, thus, more general than that of "linear space".

Clearly, a polynomial matrix is also a rational matrix, i.e. $\mathcal{P}_{m\times n} \subset \mathcal{R}_{m\times n}$. Paradoxically, as mentioned above, to analyze the set of polynomial matrices $\mathcal{P}_{m\times n}$ (a $\mathcal{P}$-module), the concepts from linear spaces are no longer adequate. Having said that, we can expect natural extensions of concepts such as rank, null space, basis, linear combination, etc. in the context of modules and submodules.

In what follows we review some results concerning the ring $\mathcal{P}$ and the $\mathcal{P}$-module $\mathcal{P}_{m\times n}$, but we also include some comments on the ring, $\mathcal{P}^t$, of polynomials in t variables $\lambda_1, \ldots, \lambda_t$, and its respective $\mathcal{P}^t$-module $\mathcal{P}^t_{m\times n}$.

We start with some basics concerning an arbitrary ring R. An *ideal* of R is a subset $J \subseteq R$ such that: $a + b \in J$, $\forall a, b \in J$, and $ap \in J$, $\forall a \in J$, $p \in R$. Any ideal J of R is clearly an R-module.

A subset $T = \{t_1, \ldots, t_k\}$ of an ideal J is a *generating set* for J if J is the set of all linear combinations over R of the elements in T, i.e. if

$$J = \Big\{ \sum_{i=1}^{k} p_i t_i \mid p_i \in R \Big\},$$

Such an ideal J is said to be *finitely generated.* If k, the minimum number of elements in the generating set of J, is equal to one, then J is a *principal ideal.* A ring R is called *Noetherian* if every ideal in R is finitely generated. A *principal ideal domain* is a ring in which every ideal is a principal ideal. It is easily verified that the ring $\mathcal{P}$ of scalar polynomials in one variable is a principal ideal domain, while the ring $\mathcal{P}^t$ is a Noetherian ring. As a consequence, it can be shown that every submodule W of $\mathcal{P}_m$ ($\mathcal{P}^t_m$) is finitely generated.

We note that linear dependence and independence can be defined in the module context just as for linear spaces. If a module W has a linearly independent generating set $T = \{t_1, \ldots, t_k\}$, then W is said to be a *free module*, and T could naturally be called a module basis of W, in which case we write $W = \langle T \rangle_{\mathcal{P}}$ ($W = \langle T \rangle_{\mathcal{P}^t}$).

Since $\mathcal{P}$ is a principal ideal domain, any submodule W of $\mathcal{P}_m$ is free. This implies that we can always find a polynomial matrix $A(\lambda) \in \mathcal{P}_{m\times n}$ of full column rank such that $W = \langle A(\lambda)\rangle_{\mathcal{P}}$. If $\mathrm{rank}(A(\lambda)) < n$, then another submodule, analogous to the null-space, can be associated with the columns of $A(\lambda)$: the *syzygy module* of $A(\lambda)$, $\mathrm{Syz}A(\lambda)$, which is the set of all polynomial vectors $z(\lambda) \in \mathcal{P}_n$ such that $A(\lambda)z(\lambda) = 0$, i.e.

$$\mathrm{Syz}A(\lambda) = \{z(\lambda) \in \mathcal{P}_n \mid A(\lambda)z(\lambda) = 0\}.$$

Moreover, if Z is a polynomial basis of $\ker A(\lambda)$, then it can be shown that

$$\langle Z\rangle_{\mathcal{P}} = \mathrm{Syz}A(\lambda) \subseteq \langle Z\rangle_{\mathcal{R}} = \ker A(\lambda), \tag{7}$$

and then it can be verified that (in terms of *algebraic* rank) $n = \mathrm{rank}(A(\lambda)) + \mathrm{rank}(\mathrm{Syz}A(\lambda))$.

From this discussion we might expect that, for all practical purposes, results concerning a polynomial matrix $A(\lambda)$ will be the same, whether $A(\lambda)$ is considered as a member of $\mathcal{P}_{m\times n}$ or of $\mathcal{R}_{m\times n}$. On the other hand, since $\mathcal{P}^t$ is Noetherian, the submodules W of $\mathcal{P}^t_m$ are in general not free, i.e. they can be generated by a linearly dependent generating set. This implies that, if Z is a polynomial basis of $\ker A(\lambda)$, then

$$\langle Z\rangle_{\mathcal{P}^t} \subseteq \mathrm{Syz}A(\lambda) \subseteq \langle Z\rangle_{\mathcal{R}^t} = \ker A(\lambda), \tag{8}$$

and $n = \mathrm{rank}(A(\lambda)) + \mathrm{rank}(\mathrm{Syz}A(\lambda))$.

Example 2.7 *Consider the ring $\mathcal{P}^3_3$ and the submodule $W = \mathrm{span}_{\mathcal{P}^3}\{z_1, z_2\}$, where*

$$z_1 = \begin{bmatrix} \lambda_3 \\ \lambda_3\lambda_2 \\ 0 \end{bmatrix}, \qquad z_2 = \begin{bmatrix} \lambda_1 \\ \lambda_1\lambda_2 \\ 0 \end{bmatrix}.$$

Clearly, the rank of $Z = [z_1, z_2]$ is one, which means that z_1 and z_2 are dependent. However, it is easy to see that there is no single polynomial $p(\lambda_1, \lambda_2, \lambda_3)$ such that $\mathrm{span}_{\mathcal{P}^3}\{p(\lambda_1, \lambda_2, \lambda_3)\} = \mathrm{span}_{\mathcal{P}^3}\{\lambda_3, \lambda_1\}$. Thus, W cannot be generated by an independent set.

A deep analysis of the case of polynomials in several variables is out of the scope of this paper. Here, we merely underline the utility of the theory of submodules when dealing with this case, see (7), (8) and the Example 2.7 above.

3 The eigenvalue structure

Part of the material of this section can be found in classical books such as [6, 7, 10]. We start with some basic definitions and results .

Definition 3.1 *Let A be a matrix with elements in a ring R. The R-*elementary operations *by rows (columns) on A are:* 1. *Interchange of two rows (columns).* 2. *Multiplication of one row (column) by a unit in R.* 3. *Adding to one row (column) an R-multiple of another row (column).*

Definition 3.2 *Matrices A and B are said to be* equivalent *if there exist nonsingular matrices U and V such that $B = UAV$.*

In the sequel we will be interested in the special case in which A and B are matrix polynomials and U and V are unimodular (or biproper) matrices. For more details about equivalent matrices and equivalence relations for polynomial matrices see Section 5.

3.1 Canonical forms for the finite eigenvalues

Lemma 3.3 *Let $U(\lambda)$ be a polynomial matrix resulting from the application of elementary operations (over the ring $\mathcal{P}$) to an identity matrix I. Then $U(\lambda)$ is unimodular.*

Unimodular matrices have no finite eigenvalues, so the equivalent matrices $A(\lambda)$ and $B(\lambda) = U(\lambda)A(\lambda)V(\lambda)$ where $U(\lambda)$ and $V(\lambda)$ are unimodular, have the same finite eigenvalue structure. In particular, we can apply $\mathcal{P}$-elementary operations by rows and columns to $A(\lambda)$ in such a way that its finite eigenvalue structure appears clearly in an equivalent matrix $B(\lambda)$.

This is in marked contrast to an eigenvalue of $A(\lambda)$ at infinity, if any. This is because multiplication of $A(\lambda)$ by an unimodular matrix polynomial can modify the structure of an eigenvalue at infinity, or create such an eigenvalue. This will be important in the next sub-section.

Theorem 3.4 *Let $A(\lambda) \in \mathcal{P}_{m\times n}$ have algebraic rank r. Then there exist unimodular matrices $U(\lambda)$ and $V(\lambda)$ such that:*

$$S^A(\lambda) = U(\lambda)A(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} f_1(\lambda) & & & \\ & \ddots & & 0 \\ & & f_r(\lambda) & \\ \hline & 0 & & 0 \end{array}\right], \tag{9}$$

where $f_1(\lambda), \ldots, f_r(\lambda)$ are uniquely defined monic polynomials and $f_i(\lambda)$ divides $f_{i+1}(\lambda)$ for $i = 1, 2, \ldots, r-1$.

Proof: The proof is by construction. At each step take the polynomial of lowest degree as a pivot in the diagonal and, with polynomial elementary operations by rows and columns reduce to zero all the elements to the right and under the pivot. (See Appendices A.1 and A.2 of [8] or [10] for more details, including a uniqueness argument.) ∎

Definition 3.5 *The diagonal matrix $S^A(\lambda)$ of* (9) *is the* Smith canonical form *of $A(\lambda)$ and the polynomials $f_i(\lambda)$ are the* invariant polynomials *of $A(\lambda)$.*

The invariant polynomials can also be characterised in the following way (see Theorem A.2.2 of [8] or [10]):

Lemma 3.6 *Let $\Delta_i(\lambda)$ be the monic greatest common divisor of all the $i \times i$ minors of $A(\lambda)$, and let $\Delta_0(\lambda) = 1$. Then the invariant polynomials of $A(\lambda)$ are given by*

$$f_i(\lambda) = \frac{\Delta_i(\lambda)}{\Delta_{i-1}(\lambda)}, \quad \text{for } i = 1, \ldots, r.$$

Definition 3.7 *Polynomials $\Delta_i(\lambda)$ are the* determinantal polynomials of *$A(\lambda)$.*

Lemma 3.6 allows us to generalize the ideas, concerning the eigenvalues of regular matrices in Section 2.1, to the general case of rectangular or singular matrix polynomials. In fact, we can say that an eigenvalue α of $A(\lambda) \in \mathcal{P}_{m\times n}$ is a value of λ where $A(\lambda)$ loses rank. The geometric multiplicity of such an eigenvalue is $m_g = r - \operatorname{rank}(A(\alpha))$. If $\operatorname{rank}(A(\lambda)) = r$ for all λ, then $A(\lambda)$ has no finite eigenvalues. In summary,

Definition 3.8 *Let $A(\lambda) \in \mathcal{P}_{m\times n}$ have algebraic rank r, and let $S^A(\lambda)$ of* (9) *be its Smith form. If α is a root of m_g different invariant polynomials with a total multiplicity m_a, then α is an eigenvalue of $A(\lambda)$ with geometric and algebraic multiplicities m_g and m_a respectively.*

Now, we generalize the idea of Jordan chains (as defined in §2.1) to the case of singular or rectangular matrix polynomials. Let α be a finite eigenvalue of $A(\lambda)$ - as in Definition 3.8. We "factor out" the part of the Smith form dependent on α in the following way: First define the $m \times n$ matrix

$$S_\alpha^A(\lambda) := \left[\begin{array}{c|c} \operatorname{diag}\left\{1, \ldots, 1, (\lambda-\alpha)^{k_1}, \ldots, (\lambda-\alpha)^{k_{m_g}}\right\} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{10}$$

where there are $r - m_g$ leading "ones" and $0 < k_1 \leq \cdots \leq k_{m_g}$. Then we may write

$$S^A(\lambda) = S_\alpha^A(\lambda) T_\alpha^A(\lambda) \tag{11}$$

for an $n \times n$ matrix

$$T_\alpha^A(\lambda) := \operatorname{diag}\left\{f_1(\lambda), \ldots, f_{r-m_g}(\lambda), \bar{f}_{r-m_g+1}(\lambda), \ldots, \bar{f}_r(\lambda), 0, \ldots, 0\right\} \tag{12}$$

where $\bar{f}_{r-m_g+1}, \ldots, \bar{f}_r$ are polynomials which do not have a zero at $\lambda = \alpha$.

Definition 3.9 *The $m \times n$ matrix $S_\alpha^A(\lambda)$ is called the* local Smith form *of $A(\lambda)$ at α. The integers $k_1, \ldots, k_{m_g}$ are the* partial multiplicities *or* structural indices *of eigenvalue α.*

Theorem 3.10 *Let $A(\lambda) \in \mathcal{P}_{m\times n}$ have algebraic rank r, and let $S^A(\lambda)$ of* (9) *be its Smith form. If α is an eigenvalue of $A(\lambda)$ with algebraic and geometric multiplicities m_a and m_g respectively, and the local Smith form at α is given by* (10)*, then there are m_g chains of generalized eigenvectors associated to α, the i-th chain contains the k_i vectors $v_{i1}, v_{i2}, \ldots, v_{ik_i}$ and $m_a = k_1 + k_2 + \cdots + k_{m_g}$. Vectors $v_{11}, v_{21}, \ldots, v_{m_g 1}$ are linearly independent.*

Proof: The number of times that α appears as a zero of the invariant polynomials of $A(\lambda)$ is clearly $\sum_{i=1}^{m_g} k_i$, so, $m_a = k_1 + k_2 + \cdots + k_{m_g}$. Now let $\bar{U}(\lambda) = U^{-1}(\lambda)$ be the inverse of $U(\lambda)$, and let $X_i(\lambda)$ denote the i-th column of a matrix $X(\lambda)$. From equations (9) to (12) we can check that

$$\begin{array}{rcl} A(\lambda)V_1(\lambda) & = & \bar{U}_1(\lambda)f_1(\lambda) \\ & \vdots & \\ A(\lambda)V_{r-m_g}(\lambda) & = & \bar{U}_{r-m_g}(\lambda)f_{r-m_g}(\lambda) \\ A(\lambda)V_{r-m_g+1}(\lambda) & = & \bar{U}_{r-m_g+1}(\lambda)(\lambda-\alpha)^{k_1}\bar{f}_{r-m_g+1}(\lambda) \\ & \vdots & \\ A(\lambda)V_r(\lambda) & = & \bar{U}_r(\lambda)(\lambda-\alpha)^{k_{m_g}}\bar{f}_r(\lambda) \\ A(\lambda)V_{r+1}(\lambda) & = & 0 \\ & \vdots & \\ A(\lambda)V_n(\lambda) & = & 0. \end{array} \tag{13}$$

When evaluated at α, the m_g linearly independent vectors $V_i(\alpha)$ for $r-m_g+1 \leq i \leq r$ are such that $A(\alpha)V_i(\alpha) = 0$, so they are eigenvectors associated with α. Now notice that if $k_i > 1$, the first derivative of $A(\lambda)V_i(\lambda)$ is also equal to zero when evaluated at α, i.e. $A'(\alpha)V_i(\alpha) + A(\alpha)V_i'(\alpha) = 0$, so $V_i'(\alpha)$ is another vector in the i-th chain of generalized eigenvectors associated with α.

Clearly, the structural indices $k_1, \ldots, k_{m_g}$ can be interpreted as the number of times that $A(\lambda)V(\lambda)$ has to be differentiated in order that column $r-m_g+i$ becomes non-zero when evaluated at $\lambda = \alpha$. Thus, each chain contains k_i generalized eigenvectors, cf. Proposition 1. For more details see [7] (Chapter S1) or [1], for instance. ∎

3.2 Canonical forms for the eigenvalue at infinity

To analyze the structure of a polynomial matrix $A(\lambda) \in \mathcal{P}_{m\times n}$ at infinity we usually analyze the structure at 0 of $\bar{A}(\lambda) = A(1/\lambda)$. However, note that $\bar{A}(\lambda)$ is in general a rational matrix. As agreed in §2.1, a first approach to analyze $\bar{A}(\lambda)$ at zero is to transform it into a polynomial matrix again (cf. equation (5)), so we can say that $A(\lambda) \in \mathcal{P}_{m\times n}$ has an eigenvalue at infinity when 0 is an eigenvalue of $A_*(\lambda) = \lambda^d \bar{A}(\lambda)$. Then, Definition 3.8 and Theorem 3.10 lead to the following structures to be associated with an eigenvalue at infinity:

Proposition 3.11 *Let $A(\lambda) \in \mathcal{P}_{m\times n}$ and let $S_0^{A_*}(\lambda)$ be the local Smith form at zero (given by (10), (11) and (12)) of the reverse polynomial $A_*(\lambda)$ (given by* (4)*). Then:*
(a) If 0 *is a zero of m_g invariant polynomials of $A_*(\lambda)$ with a total multiplicity m_a, then $A(\lambda)$ has an eigenvalue at infinity with geometric and algebraic multiplicities m_g and m_a respectively.*
(b) $A(\lambda)$ has m_g Jordan chains associated with the eigenvalue at infinity. The i-th chain consists of k_i vectors $v_{i1}, v_{i2}, \ldots, v_{ik_i}$ where $k_1 + k_2 + \cdots + k_{m_g} = m_a$ and $v_{11}, v_{21}, \ldots, v_{m_g 1}$ are linearly independent.

The integers k_i of this proposition are known as the *structural indices*, or *partial multiplicities* of the eigenvalue at infinity. (We remark that there are situations in which one would like to admit polynomials with *leading coefficients* equal to zero and, with a suitably extended definition of "degree", this situation has been examined in [14].)

The following consequence of Proposition 3.11 is easily obtained:

Corollary 3.12 *If the full rank matrix $A(\lambda) \in \mathcal{P}_{n\times n}$ has degree d and invariant polynomials $f_1, f_2, \ldots, f_n$, then the algebraic multiplicity of the eigenvalue at infinity is $nd - p$ where $p = \sum_{j=1}^{n} \delta_j$ and δ_j is the degree of f_j, $j = 1, 2, \ldots, n$.*

Proposition 3.11 clearly indicates the correlation between the eigenvalue at infinity of $A(\lambda)$ and the zero eigenvalue of $A_*(\lambda)$. But now, how can we describe the structure at infinity of $A(\lambda)$ directly from the rational matrix $\bar{A}(\lambda)$? To answer this question, we need a canonical form for matrices in $\mathcal{R}_{m\times n}$ under equivalence transformations.

Consider an arbitrary rational matrix $\bar{A}(\lambda) \in \mathcal{R}_{m\times n}$ with rank r. Let $d(\lambda)$ denote the least common multiple of the denominators of all elements of $\bar{A}(\lambda)$. Then $d(\lambda)\bar{A}(\lambda) = \hat{A}(\lambda) \in \mathcal{P}_{m\times n}$ and (using Theorem 3.4) it has a Smith form

$$S^{\hat{A}}(\lambda) = U(\lambda)(d(\lambda)\bar{A}(\lambda))V(\lambda) = \left[\begin{array}{c|c} \operatorname{diag}\{f_1(\lambda), \ldots, f_r(\lambda)\} & 0 \\ \hline 0 & 0 \end{array}\right]$$

for some unimodular $U(\lambda) \in \mathcal{P}_{m\times m}$, and $V(\lambda) \in \mathcal{P}_{n\times n}$. Thus,

$$M^{\bar{A}}(\lambda) := U(\lambda)\bar{A}(\lambda)V(\lambda) = \frac{1}{d(\lambda)}S^{\hat{A}}(\lambda) = \left[\begin{array}{c|c} \operatorname{diag}\left\{\frac{\psi_1(\lambda)}{\epsilon_1(\lambda)}, \cdots, \frac{\psi_r(\lambda)}{\epsilon_r(\lambda)}\right\} & 0 \\ \hline 0 & 0 \end{array}\right] \tag{14}$$

where, for $i = 1, 2, \ldots, r$, $\psi_i(\lambda)/\epsilon_i(\lambda) = f_i(\lambda)/d(\lambda)$ is expressed in lowest terms and, since f_i divides f_{i+1} $(i = 1, 2, \ldots, r-1)$, ψ_i divides ψ_{i+1}, *and* ϵ_{i+1} divides ϵ_i.

As with the Smith form, the diagonal matrix (14) is “essentially” unique (when all the polynomials involved are assumed to be monic, for example).

Definition 3.13 *The matrix* $M^{\bar{A}}(\lambda)$ *of* (14) *is called the* Smith-McMillan canonical form *for* $\bar{A}(\lambda) \in \mathcal{R}_{m\times n}$.

We see that the Smith-McMillan form contains explicit information on the finite zeros and poles (*the finite pole-zero structure*) of $\bar{A}(\lambda)$ in condensed form: the poles of $\bar{A}(\lambda)$ are the zeros of polynomials $\epsilon_i(\lambda)$, while the zeros of $\bar{A}(\lambda)$ are the zeros of polynomials $\psi_i(\lambda)$. Also note that a pole and a zero may occur *at the same point* (see Section 6.5.3. of [10], for example). Thus, the behaviour of $A(\lambda)$ at infinity can also be described in terms of poles and zeros at infinity - depending on the pole-zero structure at zero of $\bar{A}(\lambda) = A(1/\lambda)$. More precisely, as λ^d is the least common multiple of all the denominators in $\bar{A}(\lambda)$, from (14) and (5) we verify that

$$M_0^{\bar{A}}(\lambda) = \frac{1}{\lambda^d}S_0^{A_*}(\lambda), \tag{15}$$

and we say that the pole structure at infinity of $A(\lambda)$ is the pole structure at $s = 0$ of matrix $\bar{A}(\lambda)$ and, consequentely, the zero structure at infinity of $A(\lambda)$ is the zero structure at $s = 0$ of matrix $\bar{A}(\lambda)$ (see Example 3.18 below). Now we formalize this equivalence by giving a direct method to obtain the pole-zero structure at infinity of $A(\lambda)$ without the evaluation at $\lambda = 1/\lambda$.

Definition 3.14 *A rational function* $f(\lambda) = \frac{n(\lambda)}{d(\lambda)} \in \mathcal{R}$ *is said to be* improper *if* $\deg(n) > \deg(d)$ *and* proper *otherwise. A proper rational function is said to be* strictly proper *if* $\deg(n) < \deg(d)$ *or* biproper *if* $\deg(n) = \deg(d)$.

Since the limit when $\lambda \to \infty$ of a biproper rational function is a constant, biproper rational functions have no poles or zeros at infinity.

The scalar proper rational functions form a ring $\bar{\mathcal{R}}$, and the scalar biproper rational functions are units in this ring (have a multiplicative inverse). Similarly, the $n \times n$ proper rational matrices form a ring $\bar{\mathcal{R}}_{n\times n}$ and the units in this ring are called *biproper* or $\bar{\mathcal{R}}$*-unimodular* matrices. The determinant of a biproper matrix is clearly a biproper rational function, and so biproper matrices have no poles or zeros at infinity.

Lemma 3.15 *Let* $U(\lambda) \in \bar{\mathcal{R}}_{n\times n}$ *be a proper rational matrix resulting from the application of* $\bar{\mathcal{R}}$*-elementary operations to the* $n\times n$ *identity matrix* I*. Then* $U(\lambda)$ *is biproper.*

Biproper matrices have no poles or zeros at infinity. Consequently, if $U(\lambda)$ and $V(\lambda)$ are biproper, then the equivalent matrices $A(\lambda)$ and $B(\lambda) = U(\lambda)A(\lambda)V(\lambda)$ have the same pole-zero structure at infinity. In particular, we can apply elementary operations in $\bar{\mathcal{R}}$ to $A(\lambda)$ (by rows and columns) in such a way that its pole-zero structure at infinity appears clearly in the equivalent matrix $B(\lambda)$.

Lemma 3.16 *Let* $A(\lambda) \in \mathcal{R}_{m\times n}$ *have algebraic rank* r*. Then, there exist biproper matrices* $U(\lambda)$ *and* $V(\lambda)$ *such that:*

$$M_\infty^A(\lambda) = U(\lambda)A(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} \lambda^{-t_1} & & & \\ & \ddots & & 0 \\ & & \lambda^{-t_r} & \\ \hline & 0 & & 0 \end{array}\right],$$

where integers t_i *(possibly negative) are such that* $t_i \leq t_{i+1}$ *for* $i = 1, 2, \ldots, r-1$.

Proof: The proof is by construction. At each step take the element of $A(\lambda)$ with the lowest relative degree[2] as a pivot in the diagonal, and with elementary operations in $\bar{\mathcal{R}}$ reduce to zero all the elements to the right and under the pivot (see §6.5.3. in [10] for more details). ∎

[2] The relative degree is defined as the degree of the denominator minus the degree of the numerator of a rational function.

Definition 3.17 *Matrix $M_\infty^A(\lambda)$ is called the* Smith-MacMillan form at infinity *of $A(\lambda)$. If $t_i < 0$ there is a pole at infinity of order $|t_i|$, if $t_i > 0$ there is a zero at infinity of order t_i. The* MacMillan degree at infinity δ_∞ *of $A(\lambda)$ is defined as the number of zeros at infinity, i.e. $\delta_\infty = \sum_{i=k}^r t_i$ where k is such that $t_i > 0$ for $i \geq k$. Notice that, in practice, $M_\infty^A(\lambda) = M_0^{\bar{A}}(\lambda)|_{\lambda=1/\lambda}$.*

Example 3.18 *Consider the full rank matrix*

$$A(\lambda) = \begin{bmatrix} \lambda - 1 & \lambda^3 \\ 0 & \lambda - 1 \end{bmatrix}.$$

The dual matrix is given by

$$A_*(\lambda) = \lambda^3 A(1/\lambda) = \lambda^3 \bar{A}(\lambda) = \begin{bmatrix} \lambda^2 - \lambda^3 & 1 \\ 0 & \lambda^2 - \lambda^3 \end{bmatrix},$$

and its Smith form is

$$S^{A_*}(\lambda) = U(\lambda)A_*(\lambda)V(\lambda) = \begin{bmatrix} 1 & 0 \\ \lambda^2 - \lambda^3 & -1 \end{bmatrix} A_*(\lambda) \begin{bmatrix} 0 & 1 \\ 1 & -\lambda^2 + \lambda^3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & \lambda^4(s-1)^2 \end{bmatrix}.$$

Now, from (11) *and* (12)*, the Local Smith form at zero is given by*

$$S_0^{A_*}(\lambda) = U(\lambda)A_*(\lambda)V(\lambda)[T_0^{A_*}(\lambda)]^{-1} = U(\lambda)A_*(\lambda)\bar{V}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^4 \end{bmatrix}$$

where

$$\bar{V}(\lambda) = V(\lambda)[T_0^{A_*}(\lambda)]^{-1} = V(\lambda) \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{(\lambda-1)^2} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{\lambda^2-2\lambda+1} \\ 1 & \frac{\lambda^2}{\lambda-1} \end{bmatrix}.$$

This means that $A(\lambda)$ has an eigenvalue at infinity with multiplicities $m_g = 1$ and $m_a = 4$. We can also say that $A(\lambda)$ has four eigenvectors at infinity in a single chain. On the other hand, the local Smith-McMillan form at zero of $\bar{A}(\lambda)$ is given, as in (15), by

$$M_0^{\bar{A}}(\lambda) = U(\lambda)\frac{1}{\lambda^d}A_*(\lambda)\bar{V}(\lambda) = U(\lambda)\bar{A}(\lambda)\bar{V}(\lambda) = \begin{bmatrix} \frac{1}{\lambda^3} & 0 \\ 0 & \lambda \end{bmatrix}.$$

Now, applying directly $\bar{\mathcal{R}}$-elementary operations we obtain the Smith MacMillan form at infinity of $A(\lambda)$.

$$M_\infty^A(\lambda) = U_1(\lambda)A(\lambda)V_1(\lambda) = \begin{bmatrix} 1 & 0 \\ \frac{1}{\lambda^2-\lambda} & \frac{\lambda^2}{-\lambda^2+2\lambda-1} \end{bmatrix} A(\lambda) \begin{bmatrix} 0 & 1 \\ 1 & \frac{1-\lambda}{\lambda^3} \end{bmatrix} = \begin{bmatrix} \lambda^3 & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}.$$

Note that $M_\infty^A(\lambda) = M_0^{\bar{A}}(\lambda)|_{\lambda=1/\lambda}$. This means that three of the eigenvalues at infinity of $A(\lambda)$ are poles at infinity and only one is a zero at infinity.

Remark 3.19 *If $\bar{A}(\lambda) \in \mathcal{R}_{m\times n}$ and $\hat{A}(\lambda) = d(\lambda)\bar{A}(\lambda)$ where $d(\lambda)$ is the least common multiple of all the denominators of the elements of $\bar{A}(\lambda)$, then it may be that not all of the eigenvalues of $\hat{A}(\lambda)$ are zeros of $\bar{A}(\lambda)$. This depends on the form of $d(\lambda)$ and the partial multiplicities of each eigenvalue. In particular:*

- *A matrix polynomial is a rational matrix with no finite poles.*
- *A proper rational matrix is a rational matrix with no poles at infinity.*
- *The set of finite zeros and the set of finite eigenvalues of a polynomial matrix are the same.*

4 Subspaces and submodules

4.1 Subspaces and canonical forms

The Smith Canonical form reveals the finite eigenvalue structure of a polynomial matrix $A(\lambda)$ and, as a consequence, a polynomial basis of the kernel (null space) of $A(\lambda)$ can be computed. From (13), we observe that the last $n-r$ columns of $V(\lambda)$ form a polynomial basis for the right null space of $A(\lambda)$ (similarly, the last $m-r$ rows of $U(\lambda)$ form a polynomial basis for the left null space of $A(\lambda)$). Unfortunately, this basis is not always minimal for, as we show in the next example, it depends on the order in which the elementary operations are applied to obtain the Smith form.

Example 4.1 *Consider the following polynomial vector (which could be a row of some matrix polynomial)*

$$p(\lambda) = \begin{bmatrix} \lambda & 1+\lambda^2 & -1+\lambda \end{bmatrix}.$$

Following the procedure sketched in the proof of Theorem 3.4, we can compute the Smith canonical form as follows: as a first step, reduce the degree of the second and third elements of $p(\lambda)$:

$$p(\lambda)H_1(\lambda) = p(\lambda)\begin{bmatrix} 1 & -\lambda & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & -1 \end{bmatrix} = p_2(\lambda).$$

Then permute the columns to have a constant in the first position:

$$p_2(\lambda)H_2(\lambda) = p_2(\lambda)\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \lambda \end{bmatrix} = p_3(\lambda).$$

Finally, create zeros in the second and third positions:

$$p_3(\lambda)H_3(\lambda) = p_3(\lambda)\begin{bmatrix} 1 & 1 & -\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} = S^p(\lambda).$$

The unimodular matrix gathering all the elementary operations is

$$H(\lambda) = H_1(\lambda)H_2(\lambda)H_3(\lambda) = \begin{bmatrix} -\lambda & -1-\lambda & 1+\lambda^2 \\ 1 & 1 & -\lambda \\ 0 & 1 & 0 \end{bmatrix}.$$

The last 2 columns of $H(\lambda)$ form a polynomial basis for the kernel of $p(\lambda)$ but, clearly, this basis is not minimal as it is not column reduced. On the other hand, if we first permute columns 2 and 3 and then continue with the computation of the Smith form as before, we obtain

$$p(\lambda)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & -\lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & \lambda \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = p(\lambda)U(\lambda) = S^p(\lambda),$$

where $U(\lambda)$ is given by

$$U(\lambda) = \begin{bmatrix} 1 & -1-\lambda & 1-\lambda \\ 0 & 1 & 0 \\ -1 & 1 & \lambda \end{bmatrix}.$$

Now the last 2 columns of $U(\lambda)$ form a minimal polynomial basis for the kernel of $p(\lambda)$.

It is also clear that the columns (rows) of the Smith canonical form $S^A(\lambda)$ do not form a polynomial basis for the column (row) subspace of $A(\lambda)$. To compute polynomial basis for these subspaces, the elementary polynomial operations must be applied only by columns (rows).

Lemma 4.2 *Let $A(\lambda)$ be a polynomial matrix with algebraic rank r. Then there exists a unimodular matrix $V(\lambda)$ such that:*

$$H^A(\lambda) = A(\lambda)V(\lambda) = \left[\begin{array}{ccc|c} f_{11}(\lambda) & & & \\ \vdots & \ddots & & 0 \\ f_{r1}(\lambda) & \cdots & f_{rr}(\lambda) & \\ \hline & * & & 0 \end{array}\right], \tag{16}$$

where $*$ represents arbitrary polynomials. For $i = 1, 2, \ldots, r$, $f_{ii}(\lambda)$ are monic polynomials. For $j < i$, if deg(f_{ii}) $= 0$, then f_{ij} is zero and, if deg(f_{ii}) > 0, then deg(f_{ij}) $< \deg(f_{ii})$.

Proof: The proof is by construction (cf. the proof of Theorem 3.4). ■

Definition 4.3 (cf. §6.3.1 of [10]) *The quasi-triangular matrix $H^A(\lambda)$ of* (16) *is the* row Hermite canonical form *of $A(\lambda)$.*

Clearly, the last $n-r$ columns of $V(\lambda)$ in (16) form a polynomial basis of the kernel of $A(\lambda)$. Moreover, if S_c is the column space of $A(\lambda)$ and we define $B_c(\lambda) := \{b_1, b_2, \ldots, b_r\}$ where b_i is the ith column of $H^A(\lambda)$, then $S_c = \langle A(\lambda)\rangle_{\mathcal{R}} = \langle B_c(\lambda)\rangle_{\mathcal{R}}$.

As mentioned before (in §2.3), a polynomial basis of the kernel of a matrix $A(\lambda)$ also generates its Syzygy module. Similarly, we verify that $\langle A(\lambda)\rangle_{\mathcal{P}} = \langle B_c(\lambda)\rangle_{\mathcal{P}}$. Now, what other relations could be established between the canonical forms of $A(\lambda)$ and the submodules generated by its columns? Next, we introduce the concept of Gröbner basis and we show how this basis and the canonical forms described above are related.

4.2 Gröbner basis and submodules

The theory and tools reviewed here are material that can be found in the literature, see for example [2, 4]. One should remember that the utility of these techniques is more evident when working with polynomials in several variables. For this reason we define some concepts in this more general context. We start with some formal definitions concerning monomials and polynomials.

4.2.1 Monomials and polynomials

Definition 4.4 *A* monomial *in the n variables $\lambda_1, \lambda_2, \ldots, \lambda_n$ (for $n = 1, 2, \ldots$) is an expression of the form $\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_n^{\alpha_n}$ with $\alpha_i \in \mathbb{N}$, where $\mathbb{N}$ denotes the nonnegative integers including the zero. More briefly, we write a monomial in the form λ^γ where $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ is known as the* degree *of the monomial.*

Note that, if we conserve always the same order in the variables λ_i, any monomial could be distinguished by its degree, i.e. there is a bijection between the set $\mathcal{M}^n$ of all the monomials in n variables and $\mathbb{N}^n$. In that way, $\mathcal{M}^n$ can be ordered: we say that $\lambda^\gamma > \lambda^\eta$ if $\gamma > \eta$ for some binary relation $>$ on $\mathbb{N}^n$.

Definition 4.5 *A* monomial ordering *on $\mathcal{M}^n$ is a well-defined binary relation $>$ on $\mathbb{N}^n$. This means that, given any two degrees γ and η:*

- *only one of $\gamma > \eta$, $\gamma = \eta$ or $\eta > \gamma$ is true,*
- *if $\gamma > \eta$, then $\gamma + \beta > \eta + \beta$ for any $\beta \in \mathbb{N}^n$, and*
- *every subset of $\mathbb{N}^n$ has a smallest element.*

For the univariate case (when $n = 1$), the monomial ordering is just the classical binary relation "greater than". In contrast, when $n > 1$, we can define several different monomial orderings. The most useful are the *lexicographic ordering* ($>_{\text{lex}}$), and the *graded lexicographic ordering* ($>_{\text{glex}}$).

- $\gamma >_{\text{lex}} \eta$ if the leftmost non zero entry of $\gamma - \eta$ is positive.
- $\gamma >_{\text{glex}} \eta$ if $\sum_{i=1}^{n} \gamma_i > \sum_{i=1}^{n} \eta_i$ or, when $\sum_{i=1}^{n} \gamma_i = \sum_{i=1}^{n} \eta_i$, then $\gamma >_{\text{lex}} \eta$.

Now, with the definition of monomial ordering at a hand, we can define a polynomial without ambiguity as a linear combination (over $\mathbb{R}$) of monomials.

Definition 4.6 *A polynomial $p \in \mathcal{P}^n$ in n variables is a linear combination of the form*

$$p = \sum_{i=1}^{k} c_i \lambda^{\gamma_i}$$

where $\lambda^{\gamma_i} \in \mathcal{M}^n$, the coefficients c_i are nonzero real numbers, and $\gamma_i > \gamma_{i-1}$, $i = 2, 3, \ldots, k$ for some monomial ordering $>$. The leading monomial *of p is given by $Lm(p) = \lambda^{\gamma_k}$, then γ_k is the* degree *of p ($deg(p) = \gamma_k$). Similarly, the* leading coefficient*, and the* leading term *of p are given by $Lc(p) = c_k$ and $Lt(p) = c_k \lambda^{\gamma_k}$ respectively.*

Example 4.7 *Consider the polynomial*

$$p(\lambda_1, \lambda_2) = 5\lambda_1\lambda_2 - 7\lambda_1^3 + 4\lambda_2^2\lambda_1 - \lambda_2 + 7 + \lambda_1^2\lambda_2^2.$$

With the lexicographic ordering, f is written as the linear combination

$$p = 7 - \lambda_2 + 5\lambda_1\lambda_2 + 4\lambda_1\lambda_2^2 + \lambda_1^2\lambda_2^2 - 7\lambda_1^3,$$

and then, $Lm(p) = \lambda_1^3$, $deg(p) = (3, 0)$, $Lc(p) = -7$, and $Lt(p) = -7\lambda_1^3$. On the other hand, using a graded lexicographic ordering

$$p = 7 - \lambda_2 + 5\lambda_1\lambda_2 + 4\lambda_1\lambda_2^2 - 7\lambda_1^3 + \lambda_1^2\lambda_2^2,$$

and then, $Lm(p) = \lambda_1^2\lambda_2^2$, $deg(p) = (2, 2)$, $Lc(p) = 1$, and $Lt(p) = \lambda_1^2\lambda_2^2$. Consequently, p is a monic polynomial under this ordering.

The monomial ordering also allows us to define the concept of divisibility of monomials and polynomials:

Definition 4.8 *We say that λ^α* divides *λ^β, or λ^β is* divisible *by λ^α, if there exist a monomial λ^γ such that $\lambda^\beta = \lambda^\alpha\lambda^\gamma$. A polynomial p is* reduced *with respect to a set of polynomials $q_1, q_2, \ldots q_k$ if no monomial of p is divisible by $Lm(q_i)$ for $i = 1, 2, \ldots, k$.*

Then we can extend the idea of *long division* for polynomials in one variable to the general case (when $n > 1$).

Lemma 4.9 *Let $p, q_1, q_2, \dots q_k \in \mathcal{P}^n$. Then there exist polynomials $r, v_1, v_2 \dots, v_k \in \mathcal{P}^n$ such that $p = v_1 q_1 + \cdots + v_k q_k + r$ where r (called the* remainder *of the division of p by $q_1, q_2, \dots q_k$) is zero or is reduced with respect to $q_1, q_2, \dots q_k$.*

As we saw in the Example 4.7, the degree of a polynomial and, as a consequence, its leading term, depend on the monomial ordering used. Consequently, it is expected that the quotients and the remainder in division of polynomials depend on the choice of monomial ordering too. What is more surprising, however, is the fact that, as we show in the next example, for a given monomial ordering, the result of the division also depends on the order in which the polynomials $q_1, q_2, \dots q_k$ are taken.

Example 4.10 *Consider the polynomials*

$$p = \lambda_1^2\lambda_2 - \lambda_2^2, \quad q_1 = \lambda_1\lambda_2 - 1, \quad q_2 = \lambda_1^2, \quad q_3 = \lambda_2^2.$$

Next we illustrate the process of dividing p by q_1, q_2 and q_3. The algorithm sketched here is a natural extension of the long division algorithm. We consider a lexicographic ordering.
First note that $Lm(p) = \lambda_1^2\lambda_2$ is divisible by $Lm(q_1) = \lambda_1\lambda_2$, so we take $v_1 = \mathrm{Lt}(p)/\mathrm{Lt}(q_1) = \lambda_1$ and we produce a remainder $r = p - v_1 q_1 = \lambda_1 - \lambda_2^2$. Clearly, $Lm(r)$ is not divisible by $Lm(q_1)$, $Lm(q_2)$, or $Lm(q_3)$, so the remainder of the division will not be zero. Note however that, paradoxically, $p \in \mathrm{span}_{\mathcal{P}^2}\{q_1, q_2, q_3\}$.
On the other hand, if we divide p by q_2, q_1, q_3 (in this order), we find that the remainder is now equal to zero. In fact:

$$p = v_2 q_2 + v_3 q_3 = (\lambda_2)q_2 + (-1)q_3 = \lambda_1^2\lambda_2 - \lambda_2^2.$$

This example illustrates the fact that there could be some elements of an ideal, whose leading monomial is not divisible by the leading monomial of any of the elements in a given generator set. In the example, the leading monomial of $r = \lambda_1 - \lambda_2^2 \in \mathrm{span}_{\mathcal{P}^2}\{q_1, q_2, q_3\}$ is not divisible by $\mathrm{Lm}(q_1)$, $\mathrm{Lm}(q_2)$ or $\mathrm{Lm}(q_3)$. On the other hand, *it is by avoiding this problem that a Gröbner basis is constructed* (see the formal definition below).

Now, what happens in the case of modules? Considering that any ideal J of a ring R is also an R-module, it is not difficult to see that all the theory given above could be extended naturally to, in particular, the case of submodules of $\mathcal{P}_m^n$ - and this is the framework that interests us. Next we will show that, even for the case of m-dimensional vector polynomials in one variable ($n = 1$), the division also depends on the choice of monomial ordering and on the order in which the vectors in the division are taken. This is why a Gröbner basis results a useful tool.

Definition 4.11 *Let $B = \{v_1, v_2, \ldots, v_m\}$ be a basis of the linear space $\mathbb{R}_m$. A monomial (in the vectorial version) is an expression of the form $v_i\lambda^\gamma$, where $\lambda^\gamma \in \mathcal{M}^n$ and v_i is a vector of the basis B.*

If we define an index set $I = \{1, 2, \ldots, m\}$, then note that there is also a bijection between the set $\mathcal{M}_m^n$ of all the m-dimensional monomials and $\mathbb{N}^n \times I$. Then, we can call the pair (γ, i) the degree of a given monomial $v_i\lambda^\gamma$. Moreover, we can also define a monomial ordering for $\mathcal{M}_m^n$ analogous to the one given in Definition 4.5. For a given monomial ordering $>_*$ in $\mathcal{M}^n$, we have 2 different options for ordering $\mathcal{M}_m^n$: the *term over position ordering* ($>_{\text{top}}$), and the *position over term ordering* ($>_{\text{pot}}$).

- $(\gamma, i) >_{\text{top}} (\eta, j)$ if $\gamma >_* \eta$ or, when $\gamma = \beta$, then $i < j$.
- $(\gamma, i) >_{\text{pot}} (\eta, j)$ if $i < j$ or, when $i = j$, then $\gamma >_* \beta$.

Similarly, we can define a polynomial vector in $\mathcal{P}_m^n$ as a linear combination of monomials in $\mathcal{M}_m^n$, cf. Definition 4.6.

Example 4.12 *Consider the polynomial vector $p = \begin{bmatrix} \lambda_2^2 \\ \lambda_2 - 2\lambda_1 \end{bmatrix}$, and the lexicographic ordering in $\mathbb{N}^2$. Let e_1 and e_2 be the canonical basis of $\mathbb{R}_2$. With the top ordering, p is written as the linear combination*

$$p = \lambda_2 e_2 + \lambda_2^2 e_1 - 2\lambda_1 e_2 = \begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} \lambda_2^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -2\lambda_1 \end{bmatrix},$$

and then $Lm(p) = \lambda_1 e_2$, $deg(p) = (1, 0, 2)$, $Lc(p) = -2$, and $Lt(p) = -2\lambda_1 e_2$. On the other hand, using a pot ordering,

$$p = \lambda_2 e_2 - 2\lambda_1 e_2 + \lambda_2^2 e_1 = \begin{bmatrix} 0 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -2\lambda_1 \end{bmatrix} + \begin{bmatrix} \lambda_2^2 \\ 0 \end{bmatrix},$$

and then $Lm(p) = \lambda_2^2 e_1$, $deg(p) = (0, 2, 1)$, $Lc(p) = 1$, and $Lt(p) = \lambda_2^2 e_1$.

Finally, the concepts of divisibility and reducedness can be extended to the m-dimensional case. For the sake of clarity we rewrite Definition 4.8 and Lemma 4.9 for this case:

Definition 4.13 *We say that $v_i\lambda^\alpha$ divides $v_j\lambda^\beta$ or $v_j\lambda^\beta$ is divisible by $v_i\lambda^\alpha$ if $i = j$ and λ^α divides λ^β. A polynomial p is* reduced *with respect to a set of polynomials $q_1, q_2, \ldots q_k$ if no monomial of p is divisible by $Lm(q_i)$ for $i = 1, 2, \ldots, k$. Notice that the result of dividing two vector monomials is a scalar monomial.*

Lemma 4.14 *Let p, q_1, $q_2, \dots q_k \in \mathcal{P}_m^n$. Then there exist polynomials $r \in \mathcal{P}_m^n$, and $v_1, v_2 \dots, v_k \in \mathcal{P}^n$ such that $p = v_1 q_1 + \cdots + v_k q_k + r$ where r (called the* remainder *of the division of p by $q_1, q_2, \dots q_k$) is zero or is reduced with respect to $q_1, q_2, \dots q_k$.*

In the next example we show how the result of division depends on the order in which the polynomial vectors are taken - even for the case of polynomial vectors in one variable.

Example 4.15 *Consider the polynomials*

$$p = \begin{bmatrix} \lambda^2 + \lambda \\ 0 \\ \lambda^2 + 1 \end{bmatrix}, \quad q_1 = \begin{bmatrix} \lambda \\ 1 \\ 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} \lambda^2 \\ \lambda^2 \\ 1 \end{bmatrix}, \quad q_3 = \begin{bmatrix} \lambda \\ -\lambda \\ \lambda^2 + 1 \end{bmatrix}.$$

We illustrate the process of dividing p by q_1, q_2 and q_3. We consider a top ordering and the canonical basis for $\mathbb{R}_3$.
First note that Lm(p) $= \lambda^2 e_1$ is divisible by Lm(q_1) $= \lambda e_1$, so we take $v_1 = \mathrm{Lt}(p)/\mathrm{Lt}(q_1) = \lambda$ and we produce a remainder

$$r = p - v_1 q_1 = \begin{bmatrix} \lambda^2 + \lambda \\ 0 \\ \lambda^2 + 1 \end{bmatrix} - \lambda \begin{bmatrix} \lambda \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda \\ -\lambda \\ \lambda^2 + 1 \end{bmatrix}.$$

Clearly Lm(r) is divisible by Lm(q_3), so we take $v_3 = 1$, and finally $r = p - v_1 q_1 - v_3 q_3 = 0$. The division has remainder equal to zero which means that $p \in \mathrm{span}_{\mathcal{P}}\{q_1, q_2, q_3\}$. Now it is easy to see that, if we divide p by q_2, q_1, q_3 (in this order) then, after the first division, we produce a remainder

$$r = p - v_2 q_2 = \begin{bmatrix} \lambda^2 + \lambda \\ 0 \\ \lambda^2 + 1 \end{bmatrix} - 1 \begin{bmatrix} \lambda^2 \\ \lambda^2 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ -\lambda^2 \\ \lambda^2 \end{bmatrix},$$

whose leading monomial Lm(r) $= \lambda^2 e_2$ is not divisible by the leading monomials of q_1, q_2 or q_3. Thus, the remainder will not be zero.

Once again, we can note that there could be some elements of a submodule (in the example $\begin{bmatrix} \lambda \\ -\lambda^2 \\ \lambda^2 \end{bmatrix} \in \mathrm{span}_{\mathcal{P}}\{q_1, q_2, q_3\}$) whose leading monomial (i.e. $\lambda^2 e_2$) is not divisible by the leading monomials of the elements in some generator set (i.e. $\mathrm{Lm}(q_1) = s e_1$, $\mathrm{Lm}(q_2) = \lambda^2 e_1$, $\mathrm{Lm}(q_3) = \lambda^2 e_3$).

Next we describe a Gröbner Basis as a generator set that avoids the problem described above.

4.2.2 Gröbner Bases

The formal definition and basic properties are as follows, see [3, 4] for further details.

Definition 4.16 *Let W be an ideal in R or an R-submodule in general. A set $G = \{g_1, g_2, \ldots, g_k\} \subseteq W$ is a* Gröbner Basis *for W with respect to a given monomial ordering if, for any $w \in W$, there exists at least one element g_i such that $Lm(g_i)$ divides $Lm(w)$.*

In other words, we can show that a set $G = \{g_1, g_2, \ldots, g_k\} \subseteq W$ is a Gröbner Basis for W if and only if $\langle \mathrm{Lm}(W)\rangle_R = \langle \mathrm{Lm}(g_1), \ldots, \mathrm{Lm}(g_k)\rangle_R$, where $\mathrm{Lm}(W)$ is the set of all the leading monomials of the elements in W.

Lemma 4.17 *Let W be a R-submodule and $G = \{g_1, g_2, \ldots, g_k\} \subseteq W$ be a Gröbner basis for W. Then $W = \langle G\rangle_R$.*

The Gröbner basis property established above seems natural and is definitely to be desired. However, it is important to note that a Gröbner basis is not necessarily a linearly independent set, cf. Example 2.7. Thus, the term "basis" used here is not consistent with the usage in the context of linear spaces.

Proposition 4.18 *A set $G = \{g_1, g_2, \ldots, g_k\} \subseteq W$ is a Gröbner basis for $W \subseteq \mathcal{P}_m^n$ if and only if the remainder r on the division of any $v \in \mathcal{P}_m^n$ by the elements of G is independent of the order in which $g_1, g_2, \ldots, g_k$ are taken. Moreover, $r = 0$ for any $v \in W$.*

In the next example we verify these properties for the Gröbner basis of the submodule described in Example 4.15.

Example 4.19 *We take $Q = [q_1\ q_2\ q_3]$ as in Example 4.15, i.e.*

$$Q = \begin{bmatrix} \lambda & \lambda^2 & \lambda \\ 1 & \lambda^2 & -\lambda \\ 0 & 1 & \lambda^2+1 \end{bmatrix}.$$

Since Q is column reduced (cf. Definition 2.4), it can be shown (cf. Theorem 4.21 below) that Q is a Gröbner basis of $W = \langle Q\rangle_{\mathcal{P}}$ with respect to the top ordering and using the columns of

$$D_Q = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as a basis for $\mathbb{R}_3$. *So, taking*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

notice that the division of

$$p = \begin{bmatrix} \lambda^2 + \lambda \\ 0 \\ \lambda^2 + 1 \end{bmatrix} = v_3 + \lambda v_1 + \lambda^2 v_3 + \lambda^2 v_1 \in W$$

by Q *always has residual zero, no matter in which order the vectors* q_i *are taken. This is mainly because there is a different vector* v_i *for each leading monomial* $Lm(q_i)$, *i.e.*

$$\begin{aligned} q_1 = \begin{bmatrix} \lambda \\ 1 \\ 0 \end{bmatrix} &= v_2 - v_1 + \lambda v_1, \\ q_2 = \begin{bmatrix} \lambda^2 \\ \lambda^2 \\ 1 \end{bmatrix} &= v_3 + \lambda^2 v_2, \\ q_3 = \begin{bmatrix} \lambda \\ -\lambda \\ \lambda^2 + 1 \end{bmatrix} &= v_3 - \lambda v_2 + 2\lambda v_1 + \lambda^2 v_3, \end{aligned}$$

so clearly $\langle \mathrm{Lm}(W) \rangle_{\mathcal{P}} = \langle \mathrm{Lm}(q_1), \ldots, \mathrm{Lm}(q_3) \rangle_{\mathcal{P}}$.

Like any generating set, Gröbner bases are generally not unique. However, we can define a special Gröbner basis with such a property.

Definition 4.20 *A set* $G = \{g_1, g_2, \ldots, g_k\} \subseteq W$ *is a* reduced Gröbner basis *for* W *with respect to a given monomial ordering if it is a Gröbner basis,* g_i *is reduced with respect to* g_k *when* $k \neq i$, *and* g_i *is monic for* $i = 1, 2, \ldots, k$. *Reduced Gröbner basis are unique.*

Algorithms to compute Gröbner and reduced Gröbner basis in terms of purely algebraic manipulations (computer algebra) can be found in the literature, see [3], for example. Finally, we establish some relations between Gröbner basis and the canonical forms seen before. Alternative algorithms to compute Gröbner basis (for the univariate case at least) could be deduced from these relations.

Theorem 4.21 *Let* $A(\lambda) \in \mathcal{P}_{m \times n}$ *be a column reduced polynomial matrix. If* $m = n$, *then the columns of* $A(\lambda)$ *form a Gröbner basis for the submodule* $\langle A(\lambda) \rangle_{\mathcal{P}}$ *with respect to the top monomial ordering on* $\mathcal{P}_m$ *and using the columns of the associated matrix* D_A *(cf. Definition 2.4) as a basis for* $\mathbb{R}_m$.

Proof: To prove this result we use the description of a Gröbner basis used in Definition 4.16. Indeed, we have to prove that the leading monomial of any vector $v \in \langle A(\lambda)\rangle_{\mathcal{P}}$ is divisible by the leading monomial of, at least, one column of $A(\lambda)$. Any vector $v \in \langle A(\lambda)\rangle_{\mathcal{P}}$ can be written as a linear combination

$$v = p_1a_1 + p_2a_2 + \cdots + p_ma_m,$$

where $p_i \in \mathcal{P}$ and a_i represents the ith column of $A(\lambda)$. Write D_A in terms of its columns: $D_A = [\bar{a}_1\ \bar{a}_2 \cdots \bar{a}_m]$. Then, using the top ordering, and taking the columns $\bar{a}_i$ as a basis for $\mathbb{R}_m$, it is easy to see that $\mathrm{Lm}(a_i) = \lambda^{d_i}\bar{a}_i$. Hence, we can write

$$v = p_1(\cdots + \lambda^{d_1}\bar{a}_1) + p_2(\cdots + \lambda^{d_2}\bar{a}_2) + \cdots + p_m(\cdots + \lambda^{d_m}\bar{a}_m).$$

Now, let us suppose that $\deg(p_i) = \alpha_i$ then, clearly, $\mathrm{Lm}(v) = (\lambda^{\alpha_t}\lambda^{d_t})\bar{a}_t$ for some $t \in [1, m]$. This means (cf. Definitions 4.8 and 4.13) that the leading monomial of the arbitrary vector v is divisible by $\mathrm{Lm}(a_t)$ for some index t, so the proof is complete: The columns a_i of $A(\lambda)$ form a Gröbner basis of $\langle A(\lambda)\rangle_{\mathcal{P}}$. ∎

Definition 4.22 *(cf. §6.7.2 of [10]) A polynomial matrix $A(\lambda) \in \mathcal{P}_{m\times n}$ is in* column echelon form *or* Popov form *if it is column reduced with column degrees $d_1 \leq d_2 \leq \cdots \leq d_n$, and for each column $j = 1, 2, \ldots, n$ there exists and index k_j such that a_{k_jj} is monic of degree d_j, and:*

- *$deg(a_{ij}) < d_j$ for $i > k_j$,*
- *$deg(a_{k_ji}) < d_j$ for $i \neq j$,*
- *when $d_i = d_j$ and $i < j$ then $k_i < k_j$.*

Theorem 4.23 *Let $A(\lambda) \in \mathcal{P}_{m\times n}$ be in column echelon form. Then the columns of $A(\lambda)$ form a reduced Gröbner basis for the submodule $\langle A(\lambda)\rangle_{\mathcal{P}}$ with respect to the top monomial ordering on $\mathcal{P}_m$ and using the canonical basis for $\mathbb{R}_m$.*

Proof: The proof is similar to the one of Theorem 4.21. The fact that the Gröbner basis is reduced is a direct consequence of the conditions listed in Definition 4.22. ∎

Example 4.24 *Consider the matrix*

$$A(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & 1 \\ 1 & \lambda^2 \end{bmatrix}.$$

Clearly $A(\lambda)$ is column reduced with

$$D_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

however, the columns of D_A do not generate $\mathbb{R}_3$, so we cannot ensure that the columns of $A(\lambda)$ form a Gröbner basis of $\langle A(\lambda)\rangle_{\mathcal{P}}$ (see Theorem 4.21). On the other hand, note that $A(\lambda)$ is in echelon form and so, taking the canonical basis for $\mathbb{R}_3$, and using Theorem 4.23, we can conclude that the columns of $A(\lambda)$ form a reduced Gröbner basis of $\langle A(\lambda)\rangle_{\mathcal{P}}$ with respect to the top ordering. To illustrate that, we can write, for instance:

$$\begin{bmatrix} \lambda^3 - 1 \\ -1 \\ 0 \end{bmatrix} = -e_2 - e_1 + \lambda^3 e_1 = \lambda^2 a_1 - a_2.$$

Theorem 4.25 *Let $A(\lambda) \in \mathcal{P}_{m\times n}$ be a polynomial matrix in row Hermite form as in (16), then the columns of $A(\lambda)$ form a reduced Gröbner basis for the submodule $\langle A(\lambda)\rangle_{\mathcal{P}}$ with respect to the pot monomial ordering on $\mathcal{P}_m$ and using the canonical basis for $\mathbb{R}_m$.*

Proof: We use the same reasoning that that in the proof of Theorem 4.21. For simplicity, and without loss of generality, we assume that $\operatorname{rank}(A(\lambda)) = n$. Any vector $v \in \langle A(\lambda)\rangle_{\mathcal{P}}$ can be written as a linear combination

$$v = p_1 a_1 + p_2 a_2 + \cdots + p_n a_n,$$

where $p_i \in \mathcal{P}$ and a_i represents the ith column of $A(\lambda)$. Then, using the pot ordering, and the canonical basis $e_1, e_2, \ldots, e_m$ for $\mathbb{R}_m$, it is easy to see that $\mathrm{Lm}(a_i) = \lambda^{d_i} e_i$. So we can write

$$v = p_1(\cdots + \lambda^{d_1} e_1) + p_2(\cdots + \lambda^{d_2} e_2) + \cdots + p_n(\cdots + \lambda^{d_n} e_n).$$

Now suppose that $\deg(p_i) = \alpha_i$ then, clearly, $\mathrm{Lm}(v) = (\lambda^{\alpha_t}\lambda^{d_t})e_t$ for some $t \in [1, n]$. This means that the leading monomial of the arbitrary vector v is divisible by $\mathrm{Lm}(a_t)$ for some index t, so the columns of $A(\lambda)$ form a Gröbner basis for $\langle A(\lambda)\rangle_{\mathcal{P}}$. The "reduced" property is a consequence of the definition of the Hermite form (see Lemma 4.2). ∎

Example 4.26 *Let*

$$A(\lambda) = \begin{bmatrix} 1 & -\lambda^2 & -1+\lambda \\ 0 & 1-\lambda^4 & 1+\lambda^2 \\ \lambda & -1+\lambda+2\lambda^2-2\lambda^3 & -1+\lambda^2 \end{bmatrix}$$

be an arbitrary polynomial matrix. Considering the pot ordering and the canonical basis for $\mathbb{R}_3$, the columns $a_1(\lambda)$, $a_2(\lambda)$ and $a_3(\lambda)$ of $A(\lambda)$ do not form a Gröbner basis for $\langle A(\lambda)\rangle_{\mathcal{P}}$ since there are some vectors in $\langle A(\lambda)\rangle_{\mathcal{P}}$ whose leading monomial is not divisible by any of the leading monomials of $a_i(\lambda)$. For example, take

$$v = (1-\lambda)a_1 + a_3 = \begin{bmatrix} 0 \\ 1+\lambda^2 \\ -1+\lambda \end{bmatrix} \in \langle A(\lambda)\rangle_{\mathcal{P}}.$$

To obtain a reduced Gröbner basis for $\langle A(\lambda)\rangle_{\mathcal{P}}$ we construct the row Hermite form of $A(\lambda)$

$$\begin{aligned} H^A(\lambda) = &\begin{bmatrix} 1 & -\lambda^2 & -1+\lambda \\ 0 & 1-\lambda^4 & 1+\lambda^2 \\ \lambda & -1+\lambda+2\lambda^2-2\lambda^3 & -1+\lambda^2 \end{bmatrix} \begin{bmatrix} 1 & 1-\lambda & -1+\lambda+2\lambda^2-\lambda^3 \\ 0 & 0 & 1 \\ 0 & 1 & -1+\lambda^2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda^2+1 & 0 \\ \lambda & \lambda-1 & \lambda^2 \end{bmatrix} = \begin{bmatrix} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{bmatrix}. \end{aligned}$$

Note that now $v = \bar{a}_2$.

5 Equivalence of matrix valued functions

The concept of equivalent matrices given by Definition 3.2 defines an *equivalence relation* on $\mathcal{P}_{m\times n}$ and, consequently, divides $\mathcal{P}_{m\times n}$ into disjoint equivalence classes. In particular, we are interested in classes of matrices having the same eigenstructure. In this section we review the equivalence relations and the structure preserving transformations that have entered our discussion.

5.1 Basic equivalences

In the following definitions, $U(\lambda) \in \mathcal{P}_{m\times m}$, $V(\lambda) \in \mathcal{P}_{n\times n}$, and the reader is reminded of Definition 3.14 (proper rational functions).

Definition 5.1 *Two matrix polynomials $A(\lambda), B(\lambda) \in \mathcal{P}_{m\times n}$ are said to be*

a) Right (left) unimodular equivalent *if there exists a unimodular matrix $V(\lambda)$ ($U(\lambda)$) such that $B(\lambda) = A(\lambda)V(\lambda)$, ($B(\lambda) = U(\lambda)A(\lambda)$).*

b) Unimodular equivalent *if there exist unimodular matrices $U(\lambda)$ and $V(\lambda)$ such that $B(\lambda) = U(\lambda)A(\lambda)V(\lambda)$.*

c) Locally equivalent at $\lambda = \alpha$ *if there exist nonsingular matrices* $U(\lambda)$ *and* $V(\lambda)$ *such that* $B(\lambda) = U(\lambda)A(\lambda)V(\lambda)$, *where* $U(\lambda)$ *and* $V(\lambda)$ *are nonsingular at* $\lambda = \alpha$.
In particular, $A(\lambda)$ *and* $B(\lambda)$ *are said to be* locally equivalent at infinity *if* $B_*(\lambda) = \bar{U}(\lambda)A_*(\lambda)\bar{V}(\lambda)$ *and* $\bar{U}(\lambda)$ *and* $\bar{V}(\lambda)$ *are nonsingular at* $\lambda = 0$.

d) Strongly equivalent *if they are unimodular equivalent and locally equivalent at infinity.*

e) Biproper equivalent *if there exist biproper matrices* $U(\lambda)$ *and* $V(\lambda)$ *such that* $B(\lambda) = U(\lambda)A(\lambda)V(\lambda)$.

Some remarks concerning these definitions are in order:

- All the definitions above are equivalence relations on $\mathcal{P}_{m\times n}$.

- Two matrices $A(\lambda)$ and $B(\lambda)$ that are right unimodular equivalent generate the same linear subspace of $\mathcal{R}_n$ and hence the same submodule of $\mathcal{P}_n$, i.e $\langle A(\lambda)\rangle_{\mathcal{P}} = \langle B(\lambda)\rangle_{\mathcal{P}} \subseteq \langle B(\lambda)\rangle_{\mathcal{R}} = \langle A(\lambda)\rangle_{\mathcal{R}}$. In this case, matrices $A(\lambda)$ and $B(\lambda)$ may also be called *module equivalent*.

- Right unimodular equivalence is a special case of unimodular equivalence but not conversely.

- The row hermite form of $A(\lambda)$ (described in Theorem 4.2) can also be defined as the unique matrix $H^A(\lambda)$ in row hermite form that is module equivalent to $A(\lambda)$, i.e such that $\langle A(\lambda)\rangle_{\mathcal{P}} = \langle H^A(\lambda)\rangle_{\mathcal{P}}$. Similarly, a column reduced form of $A(\lambda)$ is any column reduced matrix $B(\lambda)$ such that $\langle A(\lambda)\rangle_{\mathcal{P}} = \langle B(\lambda)\rangle_{\mathcal{P}}$, and the column echelon form of $A(\lambda)$ is the unique matrix $B(\lambda)$ in column echelon form such that $\langle A(\lambda)\rangle_{\mathcal{P}} = \langle B(\lambda)\rangle_{\mathcal{P}}$.

- The Smith forms $S^A(\lambda)$ and $S^B(\lambda)$ are equal if and only if matrices $A(\lambda)$ and $B(\lambda)$ are unimodular equivalent. If the matrices are also strongly equivalent then $S_0^{A_*}(\lambda) = S_0^{B_*}(\lambda)$.

- Finally, $M_\infty^A(\lambda) = M_\infty^B(\lambda)$ if and only if matrices $A(\lambda)$ and $B(\lambda)$ are biproper equivalent.

For the regular case we have the following result:

Lemma 5.2 *A necessary condition for two regular matrix polynomials to be strongly equivalent is that they have the same degree. In consequence, if* $A(\lambda)$ *and* $B(\lambda)$ *are strongly equivalent, then* $M_\infty^A(\lambda) = M_\infty^B(\lambda)$.

This lemma is easily obtained from the following well known results. Consider a regular $n \times n$ polynomial matrix $A(\lambda)$ with degree d. Obviously, the degree of $\det(A(\lambda))$ does not exceed nd. So a matrix polynomial has at most nd finite eigenvalues (counting multiplicities), and this maximum occurs when $\text{rank}(A_d) = n$. If A_d is singular, then $A(\lambda)$ has an eigenvalue at infinity and:

$$nd = n_\infty + z_f \tag{17}$$

where n_∞ and z_f are, respectively, the algebraic multiplicity of the eigenvalue at infinity, and the number of finite eigenvalues (counting multiplicities) of $A(\lambda)$ (cf. Corollary 3.12). Now, suppose that

$$n_\infty = \underbrace{0 + 0 + \cdots + 0}_{\text{rank}(A_d)} + n_1 + n_2 + \cdots + n_{m_g}$$

where n_i are the partial multiplicities of the infinity, and $m_g = n - \text{rank}(A_d)$ is its geometric multiplicity, then, from equation (17),

$$(\underbrace{(d-0) + (d-0) + \cdots + (d-0)}_{\text{rank}(A_d)} + (d-n_1) + (d-n_2) + \cdots + (d-n_k) + \cdots + (d-n_{m_g}) = z_f$$

Let k be such that $n_i \geq d$ for $i = k : m_g$, then $p_\infty = z_\infty + z_f$, where $p_\infty = (d-0) + \cdots + (d-n_{k-1})$ and $z_\infty = (n_k - d) + \cdots + (n_{m_g} - d)$ are, respectively, the number of poles and zeros at infinity (as they appear in the Smith MacMillan form at infinity of $A(\lambda)$).

Note that for the case of rectangular or rank deficient matrices, equation (17) becomes

$$rd = n_\infty + z_f + \gamma_r + \gamma_l$$

where r is the rank of the matrix, and γ_r and γ_l are the sum of the degrees of the vectors in a minimal polynomial basis of the right and left null spaces respectively. For more details see section 3.6 in [26], for example.

5.2 Extended equivalence

Now we go further in the study of structure preserving transformations. In particular we would like to extend the relations proposed in Definition 5.1 to the case when matrices $A(\lambda)$ and $B(\lambda)$ are of different size. We start with some useful definitions.

Definition 5.3 *A set of polynomials $f_1, f_2, \ldots, f_n \in \mathcal{P}$ is* factor coprime *if they have no polynomial common factors.*

Definition 5.4 *Let $A(\lambda)$ and $B(\lambda)$ be two polynomial matrices with the same number of rows (columns). Any matrix $C(\lambda)$ of suitable size is a* common left (right) divisor *of $A(\lambda)$ and $B(\lambda)$ if there exist matrices $\bar{A}(\lambda)$ and $\bar{B}(\lambda)$ such that*

$$\begin{array}{ll} A(\lambda) = C(\lambda)\bar{A}(\lambda), \quad B(\lambda) = C(\lambda)\bar{B}(\lambda) & \text{(left case)}, \\ A(\lambda) = \bar{A}(\lambda)C(\lambda), \quad B(\lambda) = \bar{B}(\lambda)C(\lambda) & \text{(right case)}. \end{array}$$

Definition 5.5 *Two matrices $A(\lambda)$ and $B(\lambda)$ with the same number of rows (columns) are said to be* left (right) coprime *if all their common left (right) divisors are unimodular.*

Clearly coprimeness implies that the matrices $A(\lambda)$ and $B(\lambda)$ have no common eigenvalues, so, we can easily prove the following lemma.

Lemma 5.6 *$A(\lambda)$ and $B(\lambda)$ are left (right) coprime if and only if the compound matrices*

$$\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \quad \text{(left case)}$$
$$\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix} \quad \text{(right case)}$$

have no finite eigenvalues or, what is equivalent: their invariant polynomials are all identically equal to one, which means that the set of all their determinantal polynomials is factor coprime.

Definition 5.7 *Matrices $A(\lambda) \in \mathcal{P}_{m\times n}$ and $B(\lambda) \in \mathcal{P}_{p\times q}$ are* extended unimodular equivalent *if there exist matrices $M(\lambda)$ and $N(\lambda)$ such that*

i) $\begin{bmatrix} M(\lambda) & A(\lambda) \end{bmatrix} \begin{bmatrix} B(\lambda) \\ -N(\lambda) \end{bmatrix} = 0$, *and*

ii) $M(\lambda)$ and $A(\lambda)$ are left coprime and $-N(\lambda)$ and $B(\lambda)$ are right coprime.

Extended unimodular equivalence is an equivalence relation that allows us to work with matrices with different dimensions. As well as for the unimodular equivalence, the extended unimodular equivalence can be reformulated for the case of *extended local equivalence at $\lambda = \alpha$*, and for the case of *extended strongly equivalence*. These equivalence preserve the finite eigenvalue structure, the structure at any point $\lambda = \alpha$ and the finite and infinite eigenvalue structures respectively.

Extended strong equivalence implies two different relations: extended unimodular equivalence (with the pair of matrices $M(\lambda)$ and $N(\lambda)$), and extended local equivalence at infinity (with the, in general different, pair of matrices $\bar{M}(\lambda)$ and $\bar{N}(\lambda)$). Next we present another alternative of finite and infinite structure preserving transformations.

Definition 5.8 *Matrices $A(\lambda) \in \mathcal{P}_{m\times n}$ and $B(\lambda) \in \mathcal{P}_{p\times q}$ are* divisor equivalent *if there exist matrices $M(\lambda)$ and $N(\lambda)$ such that*

i) $A(\lambda)$ and $B(\lambda)$ are extended unimodular equivalent,

ii) $\begin{bmatrix} M(\lambda) & A(\lambda) \end{bmatrix}$ and $\begin{bmatrix} B(\lambda) \\ -N(\lambda) \end{bmatrix}$ have no eigenvalues at infinity, and

iii) $\deg \begin{bmatrix} M(\lambda) & A(\lambda) \end{bmatrix} = \deg(A(\lambda))$, *and* $\deg \begin{bmatrix} B(\lambda) \\ -N(\lambda) \end{bmatrix} = \deg(B(\lambda))$.

Lemma 5.9 *If $A(\lambda)$ and $B(\lambda)$ are divisor equivalent, then they have the same (finite and infinite) elementary divisors.*

In contrast with extended strong equivalence, note that divisior equivalence implies only one pair of matrices $M(\lambda)$ and $N(\lambda)$, however, we should also notice that divisor equivalence is an equivalence relation only when $A(\lambda)$ and $B(\lambda)$ are nonsingular [3]. In the general case, the symmetry is lost; for some examples see [11], for instance.

A linearization of a matrix polynomial $A(\lambda) \in \mathcal{P}_{n\times n}$ of degree d is a matrix pencil $L(\lambda) = L_0 + L_1\lambda \in \mathcal{P}_{nd\times nd}$ featuring the same eigenvalues with the same multiplicities. Linearizations of matrix polynomials are widely studied in the literature [7, 8]. To linearize a matrix polynomial $A(\lambda)$ is a natural way to analyze its spectral properties; in this case the analysis becomes an extension of the classical spectral theory of the real or complex pencil $L(\lambda)$. Although a review of the linearization theory is out of the scope of this survey, here we present some examples to illustrate how extended equivalence relates a matrix $A(\lambda)$ with its linearization $L(\lambda)$.

If $L(\lambda)$ is a linearization of $A(\lambda)$, then $A(\lambda)$ and $L(\lambda)$ are extended unimodular equivalent. Consider for example the matrices

$$A(\lambda) = \begin{bmatrix} 1 & 0 & 1 \\ -\lambda & \lambda & \lambda \\ \lambda+2\lambda^2 & -\lambda^2 & 2\lambda+\lambda^2 \end{bmatrix}, \qquad B(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & \lambda & 0 & 0 \\ 0 & 2 & 0 & -1 & 1+\lambda & 0 \\ 0 & 0 & 2 & 0 & 0 & \lambda \\ 0 & -1 & 3 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} \tag{18}$$

with Smith canonical forms

$$S^A(\lambda) = \operatorname{diag}\{1, \lambda, \lambda+\lambda^2\}, \qquad S^B(\lambda) = \operatorname{diag}\{I, 1, \lambda, \lambda+\lambda^2\} = \operatorname{diag}\{I, S^A(\lambda)\}.$$

This means that $A(\lambda)$ and $B(\lambda)$ share the same finite eigenvalue structure: a simple eigenvalue at $\lambda = 1$ and an semi-simple eigenvalue at $\lambda = 0$ with

[3] For this case we can also prove that condition *iii*) of Definition 5.8 is redundant.

geometric and algebraic multiplicities equal 2. The pencil $B(\lambda)$ is a linearization of $A(\lambda)$. We can also show that $A(\lambda)$ and $B(\lambda)$ are extended unimodular equivalent with matrices

$$M(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \lambda & 1 & 0 & -1 & 1-\lambda & 1 \\ -\lambda - 2\lambda^2 & 0 & 1 & 1+2\lambda & 2\lambda^2 & 2+\lambda \end{bmatrix}$$

$$N(\lambda) = \begin{bmatrix} 1 & -1 & 3 & -\frac{3}{2} & \frac{1}{2} & -1 \\ 3 & 0 & 0 & -1 & 0 & -3 \\ -1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

In fact, note that $M(\lambda)$ and $A(\lambda)$ are left coprime, $B(\lambda)$ and $N(\lambda)$ are right coprime, and

$$\begin{bmatrix} M(\lambda) & A(\lambda) \end{bmatrix} \begin{bmatrix} B(\lambda) \\ -N(\lambda) \end{bmatrix} = 0.$$

Matrices $A(\lambda)$ and $B(\lambda)$ are, however, not extended strongly equivalent since

$$S_0^{A_*}(\lambda) = \mathrm{diag}\{1, \lambda, \lambda^2\} \qquad S_0^{B_*}(\lambda) = \mathrm{diag}\{I, \lambda, \lambda, \lambda\} \neq \mathrm{diag}\{I, S_0^{A_*}(\lambda)\}.$$

A linearization that preserves also the infinite eigenvalue structure is called a *strong linearization* [14]. It is well known that the companion matrix associated to a matrix polynomial $A(\lambda)$ is a strong linearization of $A(\lambda)$.

If $L(\lambda)$ is a strong linearization of $A(\lambda)$, then $A(\lambda)$ and $L(\lambda)$ are extended strongly equivalent. Consider again the matrix $A(\lambda)$ as in (18), and the associated companion matrix

$$L(\lambda) = \begin{bmatrix} A_2 & 0 \\ 0 & I \end{bmatrix} \lambda + \begin{bmatrix} A_1 & A_0 \\ -I & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1+2\lambda & -\lambda & 2+\lambda & 0 & 0 & 0 \\ -1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & -1 & 0 & 0 & \lambda & 0 \\ 0 & 0 & -1 & 0 & 0 & \lambda \end{bmatrix}$$

that is a strong linearization of $A(\lambda)$. Notice that

$$S_0^{L_*}(\lambda) = \mathrm{diag}\{I, S_0^{A_*}(\lambda)\}, \qquad S^L(\lambda) = \mathrm{diag}\{I, S^A(\lambda)\}.$$

Matrices $A(\lambda)$ and $L(\lambda)$ are said to be extended strongly equivalent. On the one hand, with matrices

$$M(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ \lambda & 1 & 0 & -1 & 1-\lambda & 1 \\ -\lambda - 2\lambda^2 & 0 & 1 & 1+2\lambda & 2\lambda^2 & 2+\lambda \end{bmatrix}$$

$$N_L(\lambda) = \begin{bmatrix} 0 & -1 & 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

we can verify that

$$\begin{bmatrix} M(\lambda) & A(\lambda) \end{bmatrix} \begin{bmatrix} L(\lambda) \\ -N_L(\lambda) \end{bmatrix} = 0,$$

that $M(\lambda)$ and $A(\lambda)$ are left coprime, and that $L(\lambda)$ and $N_L(\lambda)$ are right coprime. On the other hand, matrices

$$\bar{M}(\lambda) = \begin{bmatrix} 1 & 0 & 0 & -\lambda & 0 & -\lambda \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$
$$\bar{N}_L(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

are such that $\bar{M}(\lambda)$ and $A_*(\lambda)$ are left coprime, $L_*(\lambda)$ and $\bar{N}_L(\lambda)$ are right coprime, and

$$\begin{bmatrix} \bar{M}(\lambda) & A_*(\lambda) \end{bmatrix} \begin{bmatrix} L_*(\lambda) \\ -\bar{N}_L(\lambda) \end{bmatrix} = 0.$$

Finally note that the fact that $L(\lambda)$ is a strong linearization of $A(\lambda)$ is not sufficient to ensure that $A(\lambda)$ and $L(\lambda)$ are divisor equivalent.

5.3 Equivalence for matrix polynomials in several variables

Now, we review some equivalence relations for matrix polynomials in several variables. In this case the concepts and terminology are less widely accepted and there are some open problems that are still to be clarified. After some general concepts, we focus on the case of two variables, with the final objective of presenting alternative definitions of finite and infinite structure preserving transformations to that provided by Definition 5.8.

First note that Definition 5.3 can be extended quite naturally to the multivariable case. A set of polynomials in t variables $f_1, f_2, \ldots, f_n \in \mathcal{P}^t$ is *factor coprime* if they have no polynomial common factors. Clearly, the question of whether $f_1, f_2, \ldots, f_n$ have common factors or not is independent on the monomial ordering we are using.

On the other hand, it is important to note that, unlike the one-variable case, for the multivariable polynomials, factor coprimeness does not imply that $f_1, f_2, \ldots, f_n$ have no common zeros. As an example, consider the polynomials $f_1(\lambda_1, \lambda_2) = 2\lambda_1^2 - \lambda_2$ and $f_2(\lambda_1, \lambda_2) = -4\lambda_1 + 2\lambda_2$ which have no polynomial common factors. However, $(\lambda_1, \lambda_2) = (1, 2)$ is a zero of both polynomials, i.e. $f_1(1, 2) = f_2(1, 2) = 0$. This leads to the following definition:

Definition 5.10 *Let $I = \langle F \rangle_{\mathcal{P}^t}$ be the ideal generated by $F = \{f_1, f_2, \ldots, f_n\}$. The* algebraic variety *V^I defined by I is the set of all the common zeros of F, i.e.*

$$V^I = \{\alpha \in \mathbb{C}^t \mid f_i(\alpha) = 0\ \forall i\}.$$

F is said to be a zero coprime set *if $V^I = \emptyset$.*

Clearly, zero coprimeness implies factor coprimeness, but the reverse is not always true as exemplified above. As a consequence, we can define two different kinds of zeros for multivariable polynomial matrices:

Definition 5.11 *Let $A(\lambda) \in \mathcal{P}^t_{m\times n}$, be a polynomial matrix in t variables and let $\Delta_i(\lambda)$ be its determinantal polynomials, i.e. the greatest common divisor of all the $i \times i$ minors. Then the ith* determinantal zeros *of $A(\lambda)$ are the zeros of polynomial $\Delta_i(\lambda)$.*

Definition 5.12 *Let I_i be the ideal generated by all the $i\times i$ minors of $A(\lambda)$. Then the ith* invariant zeros *of $A(\lambda)$ are all the elements of V^{I_i}.*

Lemma 5.13 *Let $A(\lambda)$ be a polynomial matrix in several variables with rank r. Then,*

$$I_r \subseteq I_{r-1} \subseteq \cdots \subseteq I_1,$$
$$V^{I_1} \subseteq V^{I_2} \subseteq \cdots \subseteq V^{I_r}.$$

Similarly, if we denote $D_i = \langle \Delta_i \rangle_{\mathcal{P}^t}$, then

$$D_r \subseteq D_{r-1} \subseteq \cdots \subseteq D_1,$$
$$V^{D_1} \subseteq V^{D_2} \subseteq \cdots \subseteq V^{D_r}.$$

It is a consequence of this Lemma that, if α is a ith determinantal (invariant) zero, for any $i < r$, then it is also an rth determinantal (invariant) zero. So we can simply call the rth determinantal (invariant) zeros *the* determinantal (invariant) zeros of $A(\lambda)$. It is important to note that, for the case of polynomials in one variable, the set of invariant zeros and the set of determinantal zeros are the same: the set of eigenvalues.

Now we generalize the concept of coprimeness for polynomial matrices in several variables considering the two kinds of zeros defined above.

Definition 5.14 *Two matrices $A(\lambda)$ and $B(\lambda)$ in several variables, with the same number of rows (columns) are said to be* left (right) factor coprime *if all their common left (right) divisors are unimodular, i.e. if all the matrices $C(\lambda)$ such that*

$$A(\lambda) = C(\lambda)\bar{A}(\lambda), \quad B(\lambda) = C(\lambda)\bar{B}(\lambda) \quad \text{(left case)}$$
$$A(\lambda) = \bar{A}(\lambda)C(\lambda), \quad B(\lambda) = \bar{B}(\lambda)C(\lambda) \quad \text{(right case)}$$

are unimodulars.

Definition 5.15 *Two matrices $A(\lambda)$ and $B(\lambda)$ in several variables, with the same number of rows (columns) are said to be* left (right) zero coprime *if the compound matrices*

$$\begin{bmatrix} A(\lambda) & B(\lambda) \end{bmatrix} \quad \text{(left case)}$$
$$\begin{bmatrix} A(\lambda) \\ B(\lambda) \end{bmatrix} \quad \text{(right case)}$$

have no invariant zeros or, what is equivalent, they have full rank for all $\lambda \in \mathbb{C}^t$.

Note also that for the case of one variable, zero coprimeness and factor coprimeness are both equivalent to coprimeness (see Definition 5.5 and Lemma 5.6). Now we can stablish equivalences between two polynomial matrices in several variables.

Definition 5.16 *Matrices $A(\lambda) \in \mathcal{P}^t_{m\times n}$ and $B(\lambda) \in \mathcal{P}^t_{p\times q}$ are* factor (zero) coprime equivalent *if there exist matrices $M(\lambda)$ and $N(\lambda)$ such that*

i) $\begin{bmatrix} M(\lambda) & A(\lambda) \end{bmatrix} \begin{bmatrix} B(\lambda) \\ -N(\lambda) \end{bmatrix} = 0,$

ii) $M(\lambda)$ and $A(\lambda)$ are left factor (zero) coprime and $-N(\lambda)$ and $B(\lambda)$ are right factor (zero) coprime.

For the case of one variable, factor and zero coprime equivalence are reduced to extended unimodular equivalence. Also we can show that both, factor and zero coprime equivalence preserve the elementary divisors, i.e. the determinantal zeros. Clearly, zero coprimeness implies factor coprimeness but, in addition, zero comprime equivalence preserves also the invariant zeros.

Finally we review how to apply all these results to find a finite and infinite structure preserving transformation for polynomial matrices in one variable via the homogeneous matrix.

5.3.1 Equivalence for homogeneous matrices

Definition 5.17 *Let $A(\lambda) = A_d\lambda^d + A_{d-1}\lambda^{d-1} \cdots + A_1 s + A_0$ be a polynomial matrix. The* homogeneous matrix *of $A(\lambda)$ is the two variable polynomial matrix*

$$A_{\#}(s,z) = A_d s^d + A_{d-1}s^{d-1}z \cdots + A_1 s z^{d-1} + A_0 z^d \tag{19}$$

Clearly, the finite structure of $A(\lambda)$ is given by the determinantal zeros of $A_{\#}(\lambda, 1)$, and the infinite structure of $A(\lambda)$ by the determinantal zeros at $\lambda = 0$ of $A_{\#}(1, \lambda)$.

Example 5.18 *Consider the matrix*

$$A(\lambda) = \begin{bmatrix} 1+\lambda & 0 \\ 1+3\lambda+3\lambda^2+\lambda^3 & 2+3\lambda+\lambda^2 \end{bmatrix}$$

with finite and infinite eigenvalue structures given by the canonical forms

$$S^A(\lambda) = \begin{bmatrix} 1+\lambda & 0 \\ 0 & 2+3\lambda+\lambda^2 \end{bmatrix}, \quad S_0^{A_*}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda^3 \end{bmatrix}.$$

The homogeneous form of $A(\lambda)$ is given by (as in (19))

$$A_{\#}(s,z) = \begin{bmatrix} z^3+sz^2 & 0 \\ z^3+3sz^2+3s^2z+s^3 & 2z^3+3sz^2+s^2z \end{bmatrix} = \begin{bmatrix} z^2(z+s) & 0 \\ (z+s)^3 & z(z+s)(2z+s) \end{bmatrix},$$

so, $\Delta_1(s,z) = z+s$, and $\Delta_2(s,z) = z^3(z+s)^2(2z+s)$. Then, as for the one variable case, we can write

$$S^{A_{\#}}(s,z) = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \frac{\Delta_2}{\Delta_1} \end{bmatrix} = \begin{bmatrix} s+z & 0 \\ 0 & z^3(s+z)(s+2z) \end{bmatrix}$$

from where we verify that $S^A(\lambda) = S^{A_{\#}}(\lambda,1)$ and $S_0^{A_*}(\lambda) = S_0^{A_{\#}}(1,\lambda)$.

From the example above, note that if we apply a transformation preserving the determinantal structure of the homogeneous matrix $A_{\#}(s,z)$ we are actually preserving the finite and infinite eigenvalue structure of $A(\lambda)$. This leads to the following definition:

Definition 5.19 *Matrices $A(\lambda) \in \mathcal{P}_{m\times n}$ and $B(\lambda) \in \mathcal{P}_{p\times q}$ are factor (zero) equivalent if the homogeneous matrices $A_{\#}(s,z)$ and $B_{\#}(s,z)$ are factor (zero) coprime equivalent.*

Lemma 5.20 *If $A(\lambda)$ and $B(\lambda)$ are factor equivalent, then they have the same elementary finite and infinite divisors.*

Zero equivalence becomes more important when working with matrices of several variables in general, since it preserves not only the determinantal structure, but also the invariant zeros. Finally we should note that factor equivalence is an equivalence relation only when $A(\lambda)$ and $B(\lambda)$ are nonsingular. In the general case the symmetry is not satisfied.

6 Some concluding remarks

A review of the structural properties of polynomial (and rational) matrices was presented. This kind of matrix valued functions arise naturally in several areas of applied mathematics, especially in the theory of (linear) systems and control. We focused on the theoretical aspects of structural properties that are invariant under equivalent transformations.

We have distinguished two main approaches: when the set of polynomial matrices is considered as a linear space ($\mathcal{R}_{m\times n}$) over the field of rational functions, and when the set of polynomial matrices is considered as a module ($\mathcal{P}_{m\times n}$) over the ring of polynomials.

We can define the eigenvalue structure of a polynomial matrix $A(\lambda)$ regardless whether $A(\lambda)$ is in $\mathcal{R}_{m\times n}$ or $\mathcal{P}_{m\times n}$ since, once the algebraic rank is defined, this structural information depends on the rank and null spaces of constant matrices. However, we must notice that the structure at infinity could be analyzed in two different ways: as the structure of an eigenvalue of the dual (or reverse) polynomial matrix or as the pole/zero structure at infinity of a rational matrix with no finite poles.

On the other hand, considering linear combinations over the rational functions or over the polynomials has important consequences in the way we analyze the structures related to the columns (rows) of $A(\lambda)$. Rational basis for the null space of $A(\lambda)$ (a subspace of $\mathcal{R}_n$) can always be transformed into a strictly polynomial basis -minimality is a property often required. Such polynomial basis generates also the Syzygy module of $A(\lambda)$ (a submodule of $\mathcal{P}_n$). However, for the case of multivariable polynomials this is not true in general as the ring of multivariable polynomials is Noetherian but not a principal ideal domain. A further analysis of the multivariable case was not our objective, here we merely underline the importance of the module theory for such a case.

Similarly, a polynomial basis for the column subspace of $A(\lambda)$ generates also the submodule related to the columns of $A(\lambda)$, when linear combinations are taken over $\mathcal{P}$. One interesting aspect regarding generator sets $B = \{v_1, v_2, \ldots, v_q\}$ of submodules is that, once a monomial ordering and divisibility are defined, we can verify that the residual of the division of a polynomial by B depends on the order in which the vectors v_i are taken. Gröbner and reduced Gröbner basis are generator sets that avoid this problem.

Several canonical forms have entered in our discussion. These canonical forms are extensions of the reduced forms obtained via Gaussian elimination over constant matrices. These extensions are nothing but the generalization of the elementary operations to the ring of polynomials (or rational functions). The Hermite form as well as the column reduced form of a polynomial matrix reveal us a polynomial basis for its column subspace. Moreover, we have proven that this polynomial basis is also a Gröbner basis for the corresponding sub module.

Smith and Smith MacMillan canonical forms reveal the eigenvalue structure of $A(\lambda)$ and, as equivalence relations, divide the set of all polynomial matrices into disjoint equivalent clases: the equivalent class of $A(\lambda)$ is the set of all the matrices sharing the same eigenvalue structure as $A(\lambda)$, i.e. all the matrices isospectral to $A(\lambda)$. Other structure preserving transformations were also formulated for the case where the equivalent matrices have different dimensions. These equivalence relations could be useful when analyzing the eigenvalue structure of $A(\lambda)$ via the larger matrices forming a linearization. A review of all the theory concerning linearizations was also out of the scope of this work, however, an illustrative example and basic definitions were presented in §5.2.

Finally, some insights on the structure of multivariable matrix polynomials were also given. This topic continues being developed nowadays. Our objective here was only to sketch some alternative transformations preserving the finite and infinite eigenvalue structure of $A(\lambda)$ via its homogeneous matrix.

References

[1] P.J. Antsaklis and Z. Gao, Polynomial and rational matrix interpolation: theory and control applications, Int. J. Control, vol. 58 pp. 349–404, 1993.

[2] G.D. Birkhoff and S. MacLane, A Survey of Modern Algebra Third Edition, Macmillan, New York, 1965.

[3] D. Cox, J. Little and D. O'Shea, Ideals, Varieties and Algorithms: an introduction to computational Algebraic Geometry and Commutative Algebra, Springer, New York, 1997.

[4] D. Cox, J. Little and D. O'Shea, Using Algebraic Geometry, Graduate Texts in Mathematics 185, Springer, New York, 1998.

[5] G.D. Forney, Minimal bases of rational vector spaces, with applications to multivariable linear systems, SIAM Journal on Control, vol. 13, pp. 493–520, 1975.

[6] F.R. Gantmacher, Theory of Matrices, Chelsea, New York, 1959.

[7] I. Gohberg, P. Lancaster, and L. Rodman, Matrix Polynomials, Academic Press, New York, 1982.

[8] I. Gohberg, P. Lancaster, and L. Rodman, Invariant Subspaces of Matrices with Applications, Wiley, New York, 1986 and SIAM, Philadelphia, 2006.

[9] B. Hartley and T.O. Hawkes, Rings, Modules and Linear Algebra, Chapman and Hall, London, 1970.

[10] T. Kailath, Linear Systems, Prentice Hall, Englewood Cliffs, 1980.

[11] N. Karampetakis and S. Vologiannidis, Infinite elementary divisor structure preserving transformations for polynomial matrices, Int. J. Appl. Math. Comput. Sci. vol. 13, pp. 493–503, 2003.

[12] V. Kučera, Discrete Linear Control: The Polynomial Equation Approach, John Wiley and Sons, Chichester, 1979.

[13] V. Kučera, Diophantine equations in control–a survey, Automatica, vol. 29, pp. 1361–1375, 1993.

[14] P. Lancaster, Linearization of regular matrix polynomials, Electronic Journal of Linear Algebra, vol. 17, pp. 21–27, 2008.

[15] J. J. Loiseau, Sur la modification de la structure à l'infini par un retour d'état statique, SIAM J. Control Optim., vol. 26, pp. 251–273, 1988.

[16] C.C. MacDuffee, The Theory of Matrices, Chelsea, New York, 1952.

[17] B. McMillan, Introduction to formal realizability theory, Bell System Tech. J., vol. 31, pp. 217–219, 541–600, 1952.

[18] A. S. Morse, Structural invariants of linear multivariable systems, SIAM J. Control Optim., vol. 11, pp. 446–465, 1973.

[19] W. H. L. Neven and C. Praagman, Column reduction of polynomial matrices, Linear Algebra Appl., vol. 188, pp. 569–589, 1993.

[20] J. W. Polderman and J. C. Willems, Introduction to Mathematical Systems Theory: a Behavioral Approach, Springer-Verlag, 1998.

[21] H.H. Rosenbrock, State-space and Multivariable Theory, John Wiley, New York, 1970.

[22] H.J.S. Smith, On systems of linear indeterminate equations and congruences, Philosophical Transactions of the Royal Society, vol. 151, pp. 293–326, 1861.

[23] H.J. Stetter, Numerical Polynomial Algebra, SIAM, Philadelphia, 2004.

[24] P. M. Van Dooren, The computation of Kronecker's canonical form of a singular pencil, Linear Algebra Appl. vol. 27, pp. 103–140, 1979.

[25] P. M. Van Dooren and P. Dewilde, The Eigenstructure of an Arbitrary Polynomial Matrix. Computational Aspects, Linear Algebra Appl., vol. 50, pp. 545–580, 1983.

[26] A.I.G. Vardulakis, Linear Multivariable Control, John Wiley, Chichester, UK, 1991.

[27] W.A. Wolovich, Linear Multivariable Systems, Springer Verlag, 1974.

[28] W. M. Wonham and A. S. Morse, Decoupling and pole assignment in linear multivariable systems: A geometric approach. SIAM J. Control Optim. vol. 8, pp. 1–18, 1970.

[29] J.C. Zúñiga Anaya and D. Henrion, An improved Toeplitz algorithm for polynomial matrix null-space computation, Appl. Mathematics and Comp., vol. 207, pp. 256–272, 2009.