

Application of B-Spline to Numerical Solution of a System of Singularly Perturbed Problems

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Abstract

Present paper recapitulates a numerical method based on cubic B-spline to solve boundary value problems for a system of singularly perturbed second order ordinary differential equations. The method utilizes the values of cubic B-spline and its derivatives at nodal points together with the equations of the given system and boundary conditions, ensuing into the linear matrix equation. Selected numerical examples of perturbed systems for different cases of perturbation parameters from the literature are presented, which demonstrate the efficiency of present method and also confirm how the developed algorithm is better than existing numerical methods.

Key words: Singularly perturbed problem, Cubic B-spline, Nodal points.

AMS Mathematics Subject Classification: 34D15 · 65L10

1 Introduction

Singularly perturbed boundary value problems for ordinary differential equations arise in various fields of application such as fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, gas porous electrodes theory, etc. The design and analysis of appropriate numerical methods for singularly perturbed differential equations is an area of current interest. An assortment of numerical methods has been developed for such problems.

The systems of two singularly perturbed ordinary differential equations have applications in electro-analytical chemistry. The parameters multiplying the highest derivatives characterize the diffusion coefficient of the substances. The predator-prey population dynamics also incorporate application of such systems. However, little literature pertaining to numerical solution of systems of singularly perturbed differential equations is found. Some of significant contributors are G.I. Shishkin, S. Matthews, J.J.H. Miller, E.O. Riordan, S. Bellew, N. Madden, T. Valanarasu and N. Ramanujam. [1, 2, 12-15, 18, 19].

Consider a system of two singularly perturbed ordinary differential equations of the form,

$$L^{\mathbf{r}}u(x) \equiv \begin{pmatrix} -\varepsilon_1 \frac{d^2}{dx^2} & 0 \\ 0 & -\varepsilon_2 \frac{d^2}{dx^2} \end{pmatrix} \mathbf{r}u(x) + A^{\mathbf{r}}u(x) = \mathbf{u}^{\mathbf{r}}f(x), \quad x \in D \equiv [a, b], \quad (1)$$

$$\mathbf{r}u(a) = \begin{pmatrix} p \\ r \end{pmatrix}, \quad \mathbf{r}u(b) = \begin{pmatrix} q \\ s \end{pmatrix}. \quad (2)$$

where $\varepsilon_1, \varepsilon_2$ are small positive parameters and

$$\mathbf{r}u(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad A = \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix}, \quad \mathbf{u}^{\mathbf{r}}f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$$

The functions $a_{11}(x), a_{12}(x), a_{21}(x), a_{22}(x), f_1(x), f_2(x)$ are sufficiently smooth and satisfying the inequalities

$$a_{11}(x) > |a_{12}(x)|, \quad a_{22}(x) > |a_{21}(x)|, \quad a_{12}(x) \leq 0, a_{21}(x) \leq 0 \quad \forall x \in D \equiv [a, b]. \quad (3)$$

Maximum principle and stability result of the above system can be found in [15]. For such systems three separate cases were identified in [18]: (i) $\varepsilon_1 = \varepsilon_2 = \varepsilon$, (ii) $\varepsilon_1 = \varepsilon, \varepsilon_2 = 1$ and (iii) $\varepsilon_1, \varepsilon_2$ arbitrary. The first case has been examined in [14], [19]; while [15] explains method for second case; and third case has been thrashed out in [12].

A considerable amount of work has been done for development of numerical methods for boundary value problems (BVPs) using various splines. In particular, Cubic B-spline methods represent an important class of numerical methods used for solution of boundary value problems[4], singular boundary

3 Description of method

For development of the B-spline method, consider the system (1)-(2) in the form

$$-\varepsilon_1 \frac{d^2 u_1}{dx^2} + a_{11}(x)u_1(x) + a_{12}(x)u_2(x) = f_1(x) \quad (5)$$

$$-\varepsilon_2 \frac{d^2 u_2}{dx^2} + a_{21}(x)u_1(x) + a_{22}(x)u_2(x) = f_2(x) \quad (6)$$

$$\text{where } u_1(a) = p, \quad u_1(b) = q, \quad u_2(a) = r, \quad u_2(b) = s. \quad (7)$$

$$\text{Let } u_1(x) = \sum_{j=-3}^{n-1} c_j B_j(x) \quad (8)$$

$$u_2(x) = \sum_{j=-3}^{n-1} d_j B_j(x) \quad (9)$$

be the approximate solution of given system (1-2), where c_i 's and d_i 's are unknown coefficients and $B_i(x)$'s are third degree B-Spline functions. Further, let $x_0, x_1, x_2, \dots, x_n$ be grid points in the interval $[a, b]$, so that $x_i = a + ih$ ($i = 0, 1, 2, \dots, n$); $x_0 = a, x_n = b, h = \frac{b-a}{n}$.

Now, the approximate solution given by (8) and (9) must satisfy the given system at the points $x = x_i$. For, putting values from (8) and (9) in (5-6), we get

$$-\varepsilon_1 \sum_{j=-3}^{n-1} c_j B_j''(x_i) + \sum_{j=-3}^{n-1} c_j a_{11}(x_i) B_j(x_i) + \sum_{j=-3}^{n-1} d_j a_{12}(x_i) B_j(x_i) = f_1(x_i) \quad (10)$$

$$-\varepsilon_2 \sum_{j=-3}^{n-1} d_j B_j''(x_i) + \sum_{j=-3}^{n-1} c_j a_{21}(x_i) B_j(x_i) + \sum_{j=-3}^{n-1} d_j a_{22}(x_i) B_j(x_i) = f_2(x_i) \quad (11)$$

$$i = 0, 1, 2, \dots, n$$

and boundary conditions(7) give

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \alpha_2(x_0) & \beta_2(x_0) & \gamma_2(x_0) & 0 & \dots & 0 & 0 \\ 0 & \alpha_2(x_1) & \beta_2(x_1) & \gamma_2(x_1) & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & \alpha_2(x_n) & \beta_2(x_n) & \gamma_2(x_n) \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 4 & 1 & 0 & \dots & 0 & 0 \\ \alpha_3(x_0) & \beta_3(x_0) & \gamma_3(x_0) & 0 & \dots & 0 & 0 \\ 0 & \alpha_3(x_1) & \beta_3(x_1) & \gamma_3(x_1) & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & \alpha_3(x_n) & \beta_3(x_n) & \gamma_3(x_n) \\ \cdot & \cdot & \cdot & \cdot & 1 & 4 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \alpha_4(x_0) & \beta_4(x_0) & \gamma_4(x_0) & 0 & \dots & 0 & 0 \\ 0 & \alpha_4(x_1) & \beta_4(x_1) & \gamma_4(x_1) & 0 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \dots & 0 & 0 & \alpha_4(x_n) & \beta_4(x_n) & \gamma_4(x_n) \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{bmatrix}$$

Where elements of A_i^s are given by, ($i=0,1,\dots,n$)

$$\alpha_1(x_i) = -\frac{6\varepsilon_1}{h^2} + a_{11}(x_i) = \gamma_1(x_i), \quad \alpha_2(x_i) = a_{12}(x_i) = \gamma_2(x_i),$$

$$\alpha_3(x_i) = -\frac{6\varepsilon_2}{h^2} + a_{22}(x_i) = \gamma_3(x_i) \quad \alpha_4(x_i) = a_{21}(x_i) = \gamma_4(x_i)$$

$$\beta_1(x_i) = \frac{12\varepsilon_1}{h^2} + 4a_{11}(x_i), \quad \beta_2(x_i) = 4a_{12}(x_i)$$

$$u_1(0) = u_1(1) = u_2(0) = u_2(1) = 0. \quad (23)$$

Table 2: Numerical solution of Example 1 with $\varepsilon = 10^{-4}$, $h = 10^{-5}$.

Nodes	Error in u_1		Error in u_2	
	Method by Valanarasu [19]	Present method	Method by Valanarasu [19]	Present method
0.00000	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00
0.01000	1.816519E-02	9.831942E-03	9.082561E-03	6.4652975E-04
0.02000	6.357728E-03	2.854325E-03	3.178831E-03	1.8156438E-04
0.03000	1.753178E-03	7.645328E-04	8.765580E-04	4.9843256E-05
0.04000	4.484131E-04	2.164373E-04	2.241763E-04	1.5823459E-05
0.05000	1.113958E-04	5.678757E-05	5.566825E-05	4.1654893E-06
0.06000	2.734459E-05	1.653297E-05	1.364245E-05	1.2643272E-06
0.07000	6.684015E-06	4.463937E-06	3.312210E-06	3.4326838E-07
0.08000	1.636915E-06	1.362742E-06	7.886519E-07	8.8356738E-08
0.09000	4.072917E-07	2.987543E-07	1.738455E-07	2.3573269E-08
0.10000	1.080754E-07	7.632876E-08	2.423559E-08	6.0367893E-09
0.20000	1.192100E-08	1.907643E-09	2.384182E-08	1.6154322E-09
0.30000	1.192093E-08	1.907612E-09	2.384186E-08	1.6154328E-09
0.40000	1.192093E-08	1.907612E-09	2.384186E-08	1.6154328E-09
0.50000	1.192093E-08	1.907612E-09	2.384186E-08	1.6154328E-09
0.60000	1.192093E-08	1.907612E-09	2.384186E-08	1.6154328E-09
0.70000	1.192093E-08	1.907612E-09	2.384186E-08	1.6154328E-09
0.80000	1.192100E-08	1.907643E-09	2.384182E-08	1.6154322E-09
0.90000	1.080735E-07	7.632754E-08	2.423749E-08	6.0367483E-09
0.91000	4.073021E-07	2.983928E-07	1.738351E-07	2.3573654E-08
0.92000	1.636937E-06	1.362289E-06	7.886305E-07	8.8356563E-08
0.93000	6.684060E-06	4.463937E-06	3.312165E-06	3.4326564E-07
0.94000	2.734450E-05	1.653539E-05	1.364254E-05	1.2643216E-06
0.95000	1.113947E-04	5.678729E-05	5.566937E-05	4.1654824E-06
0.96000	4.484040E-04	2.164595E-04	2.241856E-04	1.5823643E-05
0.97000	1.753225E-03	7.645328E-04	8.765102E-04	4.9843756E-05
0.98000	6.357851E-03	2.854822E-03	3.178698E-03	1.8156564E-04
0.99000	1.816542E-02	9.831628E-03	9.082279E-03	6.4652432E-04
1.00000	0.000000E+00	0.000000E+00	0.000000E+00	0.000000E+00

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