Mathematica Aeterna, Vol. 1, 2011, no. 7, 421 - 424

Two families of fractal sets $K_{J(b)}$ and $\Lambda(b)$ such that $\dim_H(K_{J(b)} \cap \Lambda(b)) > \dim_H(K_{J(b)}) \cdot \dim_H(\Lambda(b))$

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Abstract

Let $B = \{1, b, b^2, \cdots\}$ be the set of powers of b. Given two families of fractal sets as $K_{J(b)} = \Big\{ \sum_{n=1}^{\infty} i_n b^{-n} : i_n \in J(b) \subset \{0, 1, \cdots, b-1\} \Big\}$ and $\Lambda(b) = \Big\{ x \in [0,1] : |x - \frac{p}{q}| \leq \frac{1}{(b+1)q}$ for all rational $\frac{p}{q}$ with $q \in B \Big\}$. We examine to find that $\dim_H(K_{J(b)} \cap \Lambda(b)) > \dim_H(K_{J(b)}) \cdot \dim_H(\Lambda(b))$ holds for all integer $b > 1$ and $1 < Card(J(b)) < b$.

Mathematics Subject Classification: 28A80

Keywords: B-expansion; Intersection of fractal sets; Hausdorff dimension

1 Introduction

Bugeaud [1] showed that the intersection of

$$Bad_A = \Big\{ X \in [0,1]^n : \exists c(X) > 0 \ s.t. \parallel Aq - X \parallel \geq \frac{c(X)}{|q^{m/n}|}, q \in \mathbb{Z}^m \Big\}.$$

with any suitably regular fractal set E is of maximal Hausdorff dimension. This implies that

$$\dim_H(Bad_A \cap E) = \dim_H(Bad_A) \dim_H(E)$$

because $\dim_H(Bad_A) = 1$. Bugeaud also posed [2] that: it is reasonable to expect that the Hausdorff dimension of $K_{J(b)} \cap \mathcal{K}(\Psi)$ is equal to the product of the Hausdorff dimension of the set $K_{J(b)}$ and $\mathcal{K}(\Psi)$, where

$$K_{J(b)} = \Big\{ \sum_{n=1}^{\infty} \frac{i_n}{b^n} : i_n \in J(b) \subset \{0, 1, 2, \cdots, b-1\} \Big\}.$$

and

$$\mathcal{K}(\Psi) = \Big\{x \in \mathbb{R} : |x - \frac{p}{q}| \leq \Psi(q) \text{ for infinitely many rational } \frac{p}{q}\Big\}.$$

In this note, we consider two fractal sets $K_{J(b)}$ and $\Lambda(b)$, the latter is a subset of $\mathcal{K}(\Psi)$ for $\Psi(q) = \frac{1}{(b+1)q}$, which is defined by

$$\Lambda(b) = \Big\{x \in [0,1] : |x - \frac{p}{q}| \leq \frac{1}{(b+1)q} \text{ for all rational } \frac{p}{q} \text{ with } q \in B\Big\},$$

here $B = \{1, b, b^2, \cdots\}$.

We obtain that

Theorem 1.1 *Let $\Lambda(b)$ and $K_{J(b)}$ be defined as above, then*

$$\dim_H(K_{J(b)} \cap \Lambda(b)) > \dim_H(K_{J(b)}) \cdot \dim_H(\Lambda(b))$$

holds for all integer $b > 1$ and $1 < Card(J(b)) =: l < b$.

2 Proof of Theorem 1.1

We firstly analyze the structure of $\Lambda(b)$. As well known, each $x \in (0,1)$ can be expressed as the b-expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{i_n}{b^n} : i_n \in \{0, 1, 2, \cdots, b-1\}$$

If $i_n \neq 0$ and d_n is the nearest integer to xb^n, then

$$\begin{aligned} & |x - \frac{d_n}{b^n}| \leq \frac{1}{(b+1)b^n} \\ \Longleftrightarrow \quad & \frac{i_{n+1}}{b^{n+1}} + \frac{i_{n+2}}{b^{n+1}} + \cdots \leq \frac{1}{(b+1)b^n} \\ \Longleftrightarrow \quad & \frac{i_{n+1}}{b} + \frac{i_{n+2}}{b^2} + \cdots \leq \frac{1}{(b+1)} \\ \Longleftrightarrow \quad & i_{n+1} = 0 \end{aligned}$$

Thus for $x = \sum\limits_{n=1}^{\infty} \frac{i_n}{b^n}$ and all $n \geq 1$, we have

$$|x - \frac{d_n}{b^n}| \leq \frac{1}{(b+1)b^n} \Longleftrightarrow i_n \cdot i_{n+1} = 0.$$

Therefore the set $\Lambda(b)$ can be written as

$$\Lambda(b) = \Big\{ \sum_{n=1}^{\infty} \frac{i_n}{b^n} : i_n \neq 0 \Rightarrow i_{n+1} = 0 \text{ for all } n \in \mathbb{N}\Big\}. \tag{1}$$

Now we prove two lemmas as follows

Lemma 2.1 *With the set* $\Lambda(b)$ *and* $(K_{J(b)} \cap \Lambda(b))$ *defined above, we have*

$$\dim_H \Lambda(b) = \frac{\log \frac{1+\sqrt{4b-3}}{2}}{\log b}; \quad \dim_H (K_{J(b)} \cap \Lambda(b)) = \frac{\log \frac{1+\sqrt{4l-3}}{2}}{\log b}.$$

Proof. By (1), $\Lambda(b)$ is a graph directed construction with an $b \times b$ incidence matrix $A = (a_{i,j})_{i,j \le b}$, where

$$a_{i,j} = \begin{cases} 1, & \text{as } i = 0 \text{ or } j = 0; \\ 0, & \text{as } i \neq 0 \text{ and } j \neq 0. \end{cases}$$

The characteristic polynomial of A is

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{pmatrix} 1-\lambda & 1 & 1 & \cdots & 1 \\ 1 & -\lambda & 0 & \cdots & 0 \\ 1 & 0 & -\lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & -\lambda \end{pmatrix} \\ &= \det \begin{pmatrix} 1-\lambda+\frac{n-1}{\lambda} & 0 & 0 & \cdots & 0 \\ 1 & -\lambda & 0 & \cdots & 0 \\ 1 & 0 & -\lambda & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & -\lambda \end{pmatrix} \\ &= (1 - \lambda + \frac{n-1}{\lambda})\lambda^{b-1}. \end{aligned}$$

Thus the largest eigenvalue λ of A is the positive root of $1 - \lambda + \frac{n-1}{\lambda} = 0$, that is

$$\lambda = \frac{1+\sqrt{4b-3}}{2}$$

According to the method of calculation Hausdorff dimension using the Perron-Frobenius theory in [3], we get

$$\dim_H \Lambda(b) = \frac{\log \frac{1+\sqrt{4b-3}}{2}}{\log b}.$$

Next,we can easy to see that the intersection

$$K_{J(b)} \cap \Lambda(b) = \left\{ x \in K_{J(b)} : |x - \frac{p}{q}| \le \frac{1}{(b+1)q} \text{ for all rational } \frac{p}{q} \text{ with } q \in B \right\}$$

is also a graph directed construction like $\Lambda(b)$ with an $l \times l$ incidence matrix $A' = (a_{i,j})_{i,j \le l}$, where $l = Card(J(b)) < b$, and $J(b) \subset \{0, 1, 2, \cdots, b-1\}$. Then using the same method we get

$$\dim_H (K_{J(b)} \cap \Lambda(b)) = \frac{\log \frac{1+\sqrt{4l-3}}{2}}{\log b}.$$

So the lemma follows.

Lemma 2.2 $f(t) = \frac{\log \frac{1+\sqrt{4t-3}}{2}}{\log t}$ *is strictly decreasing for* $t \geq 2$.

Proof. Let $u = \frac{1+\sqrt{4b-3}}{2}$, then $b = u^2 - u + 1$. So we only need to prove that $g(u) = \frac{\log u}{\log(u^2-u+1)}$ is strictly decreasing. Since

$$g'(u) = \frac{2u-1}{\log^2(u^2-u+1)}\Big(\frac{\log(u^2-u+1)}{2u^2-u} - \frac{\log u}{u^2-u+1}\Big)$$

and when $u \geq 2$

$$\frac{\log u}{u^2-u+1} = \frac{\log u^2}{2u^2-2u+2} > \frac{\log(u^2-u+1)}{2u^2-u}.$$

So that $g'(u) < 0$ and $g(x) = \frac{\log u}{\log(u^2-u+1)}$ is strictly decreasing.

□

Proof of Theorem 1.1. For $1 < l < b$, by Lemma 2.2,

$$\frac{\frac{1+\sqrt{4b-3}}{2}}{\log b} < \frac{\frac{1+\sqrt{4l-3}}{2}}{\log l}.$$

Then

$$\frac{\frac{1+\sqrt{4b-3}}{2}}{\log b} \cdot \frac{\log l}{\log b} < \frac{\frac{1+\sqrt{4l-3}}{2}}{\log b}. \tag{2}$$

Since $K_{J(b)}$ is a self-similar set, then we have([4])

$$\dim_H(K_{J(b)}) = \frac{\log l}{\log b}. \tag{3}$$

Combining with Lemma 2.1, (2) and (3), we get Theorem 1.1.

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Received: September 18, 2011