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A Closed Form Solution for Optimal Dynamic Portfolio Problems

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Abstract

Studying the optimal dynamic portfolio problems and analyzing a Bayesian investor, who predicts the future with the past information, are developed. Adopting the martingale approach and Cameron-Martin theorem, the maximization of expected utility is converted to a system of differential equations. For the case of a given utility function, a closed-form solution of the terminal wealth is found.

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1 Introduction

There are many literatures on investigating the optimal dynamic portfolio problems, the core of which is the maximization of the investor's expected utility(see [2, 8, 9]). The solution of the utility maximization problem with full information is available under certain circumstances. For the case of complete or incomplete markets, the reader is referred to Cox and Huang[2] and Karatzas, Lehoczky and Xu[4], respectively. The special feature of the both papers is that the drift process B_t can be observed by the investor in the stock market.

Several previous works investigating the problem of the optimal dynamic portfolio in the case of unobserved drift. In the paper of Karatzas and Zhao[5], the problem of maximizing a utility function for a Bayesian investor is solved. Karatzas and Zhao proved that the optimal portfolio for the Bayesian investor exists even if the drift process cannot be observed. The reason discussed by Lakner[6] is that the investors have partial informations, where the Cameron-Martin theorem[1] is used for the computation of the terminal wealth when the Bayesian investors have optimal trading strategies. Moreover, Karatzas and Shreve[3] used Clark's formula to explicitly derive the optimal strategy. At the same time, Lakner[7] indicated this result and gave a optimal trading strategy for the Bayesian investor in the case of partial information. Lakner[7] thinks that the drift paths of Brownian motions are mere mathematical tools for model creation. Adopting martingale approach and considering only power utility functions, Rishel[10] reduced the maximization of the expected power utility function to a system of differential equations. The martingale approach to the maximization of expected utility is also addressed by Karatzas and Zhao[5] and Lakner[6, 7].

This work is denoted to the study of an utility maximization problem of the Bayesian investor who wants to maximize the expected utility from the terminal value of his portfolio on the finite time interval $[0, T]$. Motivated by the desire to extend the work made in Gady Zohar[11] in which a given Riccati differential equation is used. Comparing to Gady Zohar[11], this work uses a different Riccati equation to derive a generalized representation of $\widehat{Z}_t$, where $\widehat{Z}_t$ is an exponential martingale. Depending on the martingale approach, the maximization of the expected utility function is transformed into the system of differential equations. Finally, with the Laplace transform of $\log(\widehat{Z}_T/C)$ and a given utility function, the closed-form solution for the dynamic optimization problem is found.

This paper is organized as follows. In section 2, it recalls a model of the maximization of expected utility for Bayesian investors. We make use of the martingale approach to obtain a converted form of the model. Section 3 works out the representation of $\widehat{Z}_t$ in condition of the given Riccati equation. In section 4, the Laplace transform of $\log(\widehat{Z}_T/C)$ is computed out. In section 5, we obtain a closed-form solution for the optimal dynamic portfolio problem discussed.

2 The Model

In this section, we consider a Bayesian investor, who is given an initial wealth X_0 before entering the market and wants to maximize the expected utility of the given wealth at terminal time T.

Let's state two assumption presented in [7, 11].

Assumption 2.1 *The Bayesian investor can choose a portfolio from the market to preserve his assets. The portfolio is consisted of a stock with price S_t. The stock is assumed to satisfy*

$$dS_t = B_t S_t dt + S_t dW_t, \tag{1}$$

where the portfolio is given no interest. B_t is a drift process, which decides the *optimal portfolio and the terminal wealth.* W_t *is a normally distributed random variable with mean* b_0 *and variance* ψ_0.

Denoting by

$$dY_t = dW_t + B_t dt \tag{2}$$

and substituting (2) into (1), we know

$$dS_t = S_t dY_t. \tag{3}$$

According to the Assumption 2.1, the drift process B_t cannot be observed directly by investors. However, the wealth process is restricted to be $\mathcal{F}_t-$ adapted, where $\mathcal{F}_t$ is generated by the measurement process Y_t. So the wealth process X_t satisfies

$$X_t = X_0 + \int_0^t \pi_s dY_s, \tag{4}$$

where π_t is denoted by the portfolio process, which states the amount of money invested in the stock. In order to maximize the expected utility of the wealth process X_t, let's state an assumption on $U(x)$.

Assumption 2.2. *Let* $U(x) : (0, +\infty) \to (0, +\infty)$ *be a strictly concave, monotonically increasing utility function, which satisfies*

$$\lim_{x \to 0^+} U'(x) = +\infty, \lim_{x \to \infty^-} U'(x) = 0.$$

Our model is to get the value function and the wealth process X_t. The value function is defined as

$$\nu(X_0) = \max_{\pi_t \in \Im(X_0)} E[U(X_T)], \tag{5}$$

where $\Im(X)$={All process π_t adapted to $\mathcal{F}_t$ s.t. $\int_0^T \pi_s ds < +\infty$ a.s and $X_t \geq 0$ a.s, $\forall (t) \in [0, T]$}.

As mentioned above, this problem is ill-posed because the drift B_t cannot be calculated accurately. In other words, the maximization of expected utility cannot be obtained from the calculation of the drift B_t. For this reason, we employ the martingale approach to deal with this problem(see Karatzas and Zhao [5]). As defined in [11], we write

$$Z_t = \exp\Big(- \int_0^t B_s^2 dY_s - \frac{1}{2}\int_0^t B_s^2 ds \Big), \tag{6}$$

where Y_t is a $\widetilde{P}-$ Brownian motion, independent of B_t. $\widetilde{P}(A) := E[Z_T \, \|_A]$, $0 \leq t \leq T$. We write

$$\widehat{Z}_t = E[Z_T \mid \mathcal{F}_t]. \tag{7}$$

By the Girsanov Theorem(see Karatzas and Shreve [3]), X_t is a $\widetilde{P}-$ local martingale derived from the equation (4) and $E[X_{t+s} \mid \mathcal{F}_t] = X_t$. Obviously,

$X_t\widehat{Z}_t$ is a $\widetilde{P}-$ local martingale and $E[X_T\widehat{Z}_T] \geq X_0$. The identification of the wealth process X_t is denoted by

$$X_t = \widetilde{E}[X_T \mid \mathcal{F}_t]. \tag{8}$$

For any non-negative τ, we have

$$\begin{aligned} U(X_T) &\leq EU(X_T) + \tau(X_0 - E(X_T\widehat{Z}_T)) \\ &= \tau X_0 + E[U(X_T) - \tau X_T\widehat{Z}_T] \\ &\leq \tau X_0 + E[\sup(U(X_t) - \tau X_t\widehat{Z}_T)]. \end{aligned} \tag{9}$$

So the optimal dynamic portfolio problem is transformed to maximize the right-hand side of (9). We denote

$$F(x) = U(x) - \tau x\widehat{Z}_T. \tag{10}$$

There must exists a feasible strategy for the investor. Therefore, we can find x satisfying $F'(x) = 0$, which makes $U'(x) = \tau Z_T$ be valid. Denoting $I(\cdot)$ as the inverse of $U'(\cdot)$ and replacing X_T with x, we obtain $X_T = I(\tau\widehat{Z}_T)$. Using $\widetilde{E}[X_t] = E[\widehat{Z}_tX_t]$ and $E[X_T\widehat{Z}_T] = X_0$, we get

$$X_0 = E[\widehat{Z}_T I(\tau\widehat{Z}_T)] = \widetilde{E}[I(\tau\widehat{Z}_T)]. \tag{11}$$

Thus $\tau \equiv \tau(X_0)$. The model (5) is converted into

$$\max_{\pi_t \in \Im(X_0)} E[U(X_T)] = E[U \circ I(\tau(X_0)\widehat{Z}_T)], \tag{12}$$

where $U \circ I(\cdot) = U(I(\cdot))$. Furthermore, we obtain

$$X_t = \widetilde{E}[I(\tau(X_0)\widehat{Z}_T) \mid \mathcal{F}_t]. \tag{13}$$

For computing the value function and the wealth process, our target is to calculate the representation of $\widehat{Z}_T$ in next section. Usually, the Monte Carlo simulations can be used for the question. However, it is so complexity that we pick the dynamic programming approach to transform the maximization of expected utility into a system of differential equations(see Rishel [10]).

3 Computing $\widehat{Z}_T$ in condition of the Riccati Equation

In this section, we deal with the computation of $\widehat{Z}_T$. On the basis of the Kalman filter theory and General Bayesian formula, we know $\widehat{Z}_t = E[\xi\widehat{Z}_t \mid \mathcal{F}_t]$, $\xi \in \mathcal{F}_t$ and derive

$$\begin{aligned} E[\xi(\widehat{Z}_t - E[\widehat{Z}_t \mid \mathcal{F}_t])] &= \widetilde{E}[\widetilde{E}[Z_t^{-1}\xi(\widehat{Z}_t - E[\widehat{Z}_t \mid \mathcal{F}_t]) \mid \mathcal{F}_t]] \\ &= \widetilde{E}[\xi(\widetilde{E}[Z_t^{-1}\widehat{Z}_t \mid \mathcal{F}_t] - \widetilde{E}[Z_t \mid \mathcal{F}_t]E[\widehat{Z}_t \mid \mathcal{F}_t] = 0, \end{aligned}$$

form which we have

$$Z_t = (E[Z_t^{-1} \mid \mathcal{F}_t])^{-1}. \tag{14}$$

Let $\widehat{B}_t = E[B_t \mid \mathcal{F}_t]$ be a Gaussian process, which indicates that the Bayesian investor uses the past information to predict the present number. By the theorem proved by Liptser and Shiryayev [8], $\widehat{Z}_t^{-1}$ is a $\widetilde{P}-$ martingale and satisfies

$$\widehat{Z}_t^{-1} = 1 + \int_0^t \widehat{Z}_s^{-1} B_s dY_s. \tag{15}$$

Taking conditional expectation for (15) gives rise to

$$\begin{aligned}
\widehat{Z}_t^{-1} &= 1 + \widetilde{E}\Big[\int_0^t \widehat{Z}_s^{-1} B_s dY_s \mid Y_0^t\Big] \\
&= 1 + \int_0^t \widetilde{E}[\widehat{Z}_s^{-1} B_s \mid Y_0^s] dY_s, = 1 + \int_0^t \widetilde{E}[\widehat{Z}_s^{-1} \mid Y_0^s] E[B_s \mid Y_0^s] dY_s \\
&= 1 + \int_0^t \widehat{Z}_s^{-1} \widehat{B}_s dY_s.
\end{aligned}$$

Solving the Stochastic Differential Equation (SDE) yields

$$\widehat{Z}_t = \exp\Big(- \int_0^t \widehat{B}_s dY_s + \frac{1}{2}\int_0^t \widehat{B}_s^2 ds\Big). \tag{16}$$

By Kalman filter, it exploits the fact that $\widehat{B}_t$ satisfies the following formula(see Laker[6])

$$\widehat{B}_t = \phi_t\Big(b_0 + \int_0^t \frac{\psi_s}{\phi_s} dY_s + \mu \int_0^t \frac{1}{\phi_s} ds\Big), \tag{17}$$

where $\phi_t = \exp\big(- \beta t - \int_0^t \psi_s ds\big)$. To solve the accurately representation of $\widehat{Z}_t$, we design ψ_t to satisfy the following Riccati differential equation

$$\frac{d\psi_t}{dt} = k\psi_t^2 - 2\beta\psi_t + \sigma^2 \tag{18}$$

with the initial condition ψ_0 and denote $\Delta = k\sigma^2 - \beta^2$.

There are three cases for the solution of the equation (18).

1° $\Delta = k\sigma^2 - \beta^2 < 0$,

$$\psi_t = \frac{\beta}{k} - \frac{\sqrt{-\Delta}}{k}\tanh((t + C_1)\sqrt{-\Delta}), C_1 = \frac{-1}{\sqrt{-\Delta}}\text{artanh}\frac{k\psi_0 - \beta}{\sqrt{-\Delta}}. \tag{19}$$

2° $\Delta = k\sigma^2 - \beta^2 = 0$,

$$\psi_t = \frac{\beta}{k} - \frac{1}{k(t + C_2)}, C_2 = \frac{-1}{k\psi_0 - \beta}. \tag{20}$$

3° $\Delta = k\sigma^2 - \beta^2 > 0$,

$$\psi_t = \frac{\beta}{k} + \frac{\sqrt{\Delta}}{k}\tanh((t+C_3)\sqrt{\Delta}), C_3 = \frac{1}{\sqrt{\Delta}}\text{arcot}\frac{k\psi_0 - \beta}{\sqrt{\Delta}}. \tag{21}$$

Remark 1. The Riccati differential equation (18) is different from the equation used in Gady Zhohar[11].

Now, let's compute $\widehat{Z}_t$. Firstly, let's deal with the first term on the right hand side of (16)

$$\int_0^t \widehat{B}_s dY_s = \int_0^t \Big(\Phi_s b_0 + \phi_s \mu \int_0^s \frac{1}{\phi_u}\Big) dY_s + \int_0^t \Big(\phi_s \int_0^s \frac{\psi_u}{\phi_u} dY_u\Big) dY_s. \tag{22}$$

The last term of equation (22) can be solved by the Ito' Lemma. Denoting by $f(Y_s) = \big(\int_0^s \frac{\psi_u}{\phi_u} dY_u\big)^2$, we have

$$f'(Y_s)dY_s = df(Y_t) - \frac{1}{2}f''(Y_s)ds.$$

Multiplied by $\phi_s^2/(2\psi_s)$, the last term of (22) is

$$\begin{aligned}\int_0^t \Big(\phi_s \int_0^s \frac{\psi\psi_u}{\phi_u} dY_u\Big) dY_s &= \int_0^t \frac{\phi_s^2}{2\psi_s} d\Big(\int_0^s \frac{\psi_u}{\phi_u}\Big)^2 - \frac{1}{2}\int_0^t \psi_s ds \\ &= \frac{\phi_t^2}{2\psi_t}\Big(\int_0^s \frac{\psi_u}{\phi_u} dY_u\Big)^2 - \frac{1}{2}\int_0^t \Big(\int_0^s \frac{\psi_u}{\phi_u} dY_u\Big)^2 d\Big(\frac{\phi_s^2}{\psi_s}\Big) - \frac{1}{2}\int_0^t \psi_s ds. \end{aligned} \tag{23}$$

Secondly, the last component of (16) is written as

$$\begin{aligned}\frac{1}{2}\int_0^t \widehat{B}_s^2 ds &= \frac{1}{2}\int_0^t \phi_s^2\Big(b_0 + \mu\int_0^s \frac{1}{\phi_u} du + \int_0^s \frac{\psi_u}{\phi_u} dY_u\Big)^2 ds \\ &= \frac{1}{2}\int_0^t \phi_s^2\Big(b_0 + \mu\int_0^s \frac{1}{\phi_u} du\Big)^2 + \frac{1}{2}\int_0^t \phi_s^2\Big(\int_0^s \frac{\psi_u}{\phi_u} dY_u\Big)^2 ds \\ &\quad + \int_0^t \phi_s^2\Big(b_0 + \mu\int_0^s \frac{1}{\phi_u} du\Big)\Big(\int_0^s \frac{\psi_u}{\phi_u} dY_u\Big) ds. \end{aligned} \tag{24}$$

The equations (22) and (24) are so complex that they need to be simplified. Thus, we define

$$f_s = \frac{\psi_s}{\phi_x}, F(Y_s^t) = \int_s^t f_u dY_u, \tag{25}$$

$$g_s = -\phi_s\Big(b_0 + \mu\int_0^s \frac{1}{\phi_u} du\Big), G(Y_s^t) = \int_s^t g_u dY_u, \tag{26}$$

$$p_s = -\frac{\sigma^2}{2f_s^2}, q_s = -\phi_s g_s, h_s = -\frac{1}{2}\phi_s^2, \tag{27}$$

$$A = -\frac{\phi_T^2}{2\psi_T}, \tag{28}$$

$$C = \exp\Big(\frac{1}{2}\int_0^T (g_s^2 + \psi_s)ds\Big). \tag{29}$$

We now pack (25), (26), (27), (28) and (29) back into (22), (23) and (24), and substitute (22) and (24) into (16), the representation of $\widehat{Z}_t$ is written as

$$\widehat{Z}_t = C\exp\Big(G(Y_0^t) + AF^2(Y_0^t) + \frac{1}{2}\int_0^t F^2(Y_0^s)d\Big(\frac{\phi_s^2}{\psi_s}\Big) + \int_0^t q_sF(Y_0^s)ds + \frac{1}{2}\int_0^t \phi_s^2F^2(Y_0^s)ds\Big). \tag{30}$$

Depending on the Riccati differential equation (18), we can find

$$\frac{d}{ds}\Big(\frac{\phi_s^2}{\psi_s}\Big) = \frac{d}{ds}\Big(\frac{\exp\big(-2\beta s - \int_0^s 2\psi_u du\big)}{\psi_s}\Big) = \phi_s^2\Big(\frac{-2\beta\psi_s - 2\psi_s^2 - d\psi_s/ds}{\psi_s}\Big) = -\frac{\sigma^2\phi_s^2}{\psi_s^2} - (k+2)\phi_s^2. \tag{31}$$

Substituting (31) into (30), the final representation of $\widehat{Z}_t$ becomes

$$\widehat{Z}_t = C\exp\Big(AF^2(Y_0^t) + \int_0^t (p_s + (k+1)h_s)F^2(Y_0^s)ds + G(Y_0^t) + \int_0^t q_sF(Y_0^s)ds\Big). \tag{32}$$

Remark 2. It is formula (32) that we address in the section. The $\widehat{Z}_t$ is especially useful for computing the wealth process and the value function. There are three cases about $\widehat{Z}_t$ since three cases for the solution of the Riccati differential equation (18) have been given.

4 The Laplace transform of $\log(\widehat{Z}_T/C)$

In this section, we apply the Laplace transform to compute the expectations as expressed in formulas in (11), (12) and (13). The Cameron-Martin theorem is helpful for the question (see Cameron Martion[1]). Now, we state the theorem.

Theorem 4.1 *Denoting that T is a fixed time, $A, C, \lambda \in R$. Let f_t, g_t, p_t and q_t are continuous integrable functions in interval $[0,T]$. The boundary of f_t is away from zero in $[0,T]$. Then,*

$$L(\lambda) = E\Big[e^{-\lambda\log(\widehat{Z}_T/C)}\Big] = \Big(\frac{u_T}{u_0}\Big)^{\frac{1}{2}} e^{\frac{1}{2}\lambda^2\rho^2} \tag{33}$$

which is equivalent to

$$
\begin{aligned}
&E\Big[e^{-\lambda(AF^2(Y_0^T)+\int_0^T(p_t+(k+1)h_t)F^2(Y_0^t)dt)}\Psi\Big(G(Y_0^T)+\int_0^T q_tF(Y_0^t)dt\Big)\Big]\\
&=E\Big[e^{-\lambda\big(AF^2(Y_0^T)+\int_0^T(p_t+(k+1)h_t)F^2(Y_0^t)dt+G(Y_0^T)+\int_0^T q_tF(Y_0^t)dt\big)}\Big]\\
&=\Big(\frac{u_T}{u_0}\Big)e^{\frac{1}{2}\lambda^2\rho^2},
\end{aligned}
\tag{34}
$$

where $\Psi(\cdot)$ *is an exponential function, the integral of which is finite.* u_t *is any non-trivial solution of the following ordinary differential equation*

$$
\frac{d^2u_t}{dt^2}-2\frac{f_t'du_t}{f_tdt}-2\lambda(p_t+(k+1)h_t)f_t^2u_t=0,
\tag{35}
$$

which satisfies the boundary condition

$$
\frac{d(\log u_t)}{dt}\Big|_{t=T}=-2A\lambda f_t^2
\tag{36}
$$

and

$$
\rho^2=\int_0^T m_t^2dt, m_t=\frac{f_t}{u_t}\int_t^T\Big(\frac{g_su_s'}{f_s}+q_su_s\Big)ds+g_s.
\tag{37}
$$

Proof: Before proving this theorem, we specialize a simple case, which does not consider $\Psi(\cdot)$. So we come up with an idea to find an appropriate function l_t, which makes $E[\exp(\frac{1}{2}\int_0^t l_s^2F^2(Y_0^s)ds)]<\infty$. Let

$$
\Lambda_t=\exp\Big(\int_0^t l_sF(Y_0^s)ds-\frac{1}{2}\int_0^t l_s^2F^2(Y_0^s)ds\Big)
\tag{38}
$$

be a exponential martingale, which is the multiplicative non-random function. From the martingale property, we know $E[\Lambda_T]=1$.

Applying the same method to derive equation (22), we use Ito' Lemma for the first term in the right-hand side of (38). We get

$$
\begin{aligned}
E[\Lambda_T]&=E\Big[\exp\Big(\int_0^T l_t\Big(\frac{1}{2f_t}dF^2(Y_0^t)-\frac{f_t}{2}dt\Big)-\frac{1}{2}\int_0^T l_t^2 2F^2(Y_0^t)dt\Big)\Big]\\
&=E\Big[\exp\Big(\int_0^T\frac{l_t}{2f_t}dF^2(Y_0^t)-\frac{1}{2}\int_0^T(l_tf_t+l_s^2F^2(Y_0^t))\Big)\Big]\\
&=E\Big[\exp\Big(\frac{l_T}{2f_T}F^2(Y_0^T)-\frac{1}{2}\int_0^T F^2(Y_0^t)\Big(\frac{l_t'f_t-f_t'l_t}{f_t^2}\Big)dt\\
&-\frac{1}{2}\int_0^T(l_tf_t+l_s^2F^2(Y_0^t))dt\Big)\Big].
\end{aligned}
$$

Therefore, we have

$$
E\Big[\exp\Big(\frac{l_T}{2f_T}F^2(Y_0^T)-\frac{1}{2}\int_0^T F^2(Y_0^t)\Big(\frac{l_t'f_t-f_t'l_t}{f_t^2}+l_t^2\Big)dt\Big)\Big]=\exp\Big(\frac{1}{2}\int_0^T l_tf_tdt\Big).
\tag{39}
$$

We know that u_t is the non-trivial solution of the ODE (35). Denoting by $l_t = \frac{u_t'}{u_t f_t}$, Equation (39) becomes

$$E\Big[\exp\Big(\frac{u_t'}{2u_t f_t}F^2(Y_0^T) - \int_0^T F^2(Y_0^T)\Big(\frac{u_t'' f_t - 2u_t'\frac{f_t'}{f_t}}{2u_t f_t^2}\Big)dt\Big)\Big] = \exp\Big(\frac{1}{2}\int_0^T \frac{u_t'}{u_t}dt\Big). \tag{40}$$

Substituting (35) and (36) into (40), we obtain

$$E\Big[e^{-\lambda\big(AF^2(Y_0^T)+\int_0^T (p_t+(k+1)h_t)F^2(Y_0^t)dt\big)}\Big] = \Big(\frac{u_T}{u_0}\Big)^{\frac{1}{2}}. \tag{41}$$

Now, we add $\Psi(\cdot)$ into the model (41). From the change of probability measure, we know $E_{\widetilde{P}}[X] = E_P[\Lambda X]$ and define

$$\begin{aligned} L(\lambda) &= E\Big[e^{-\lambda(AF^2(Y_0^T)+\int_0^T (p_t+(k+1)h_t)F^2(Y_0^t)dt)}\Psi\Big(G(Y_0^T) + \int_0^T q_t F(Y_0^t)dt\Big)\Big] \\ &= \Big(\frac{u_T}{u_0}\Big)^{\frac{1}{2}} E\Big[\Lambda_T \Psi\Big(G(Y_0^T) + \int_0^T q_t F(Y_0^t)dt\Big)\Big] \\ &= \Big(\frac{u_T}{u_0}\Big)^{\frac{1}{2}} \widetilde{E}\Big[\Psi\Big(G(Y_0^T) + \int_0^T q_t F(Y_0^t)dt\Big)\Big], \end{aligned} \tag{42}$$

where $\{\widetilde{Y}_t\}_{t\geq 0}$ is a standard Brownian motion defined by $d\widetilde{Y}_t = dY_t - l_t F(Y_0^t)dt$. $\{l_t F(Y_0^t)\}_{t\geq 0}$ is an $\{\widetilde{\mathcal{F}}_t\}-$adapted process since $E[\exp(\frac{1}{2}\int_0^t l_s^2 F^2(Y_0^s)ds)] < \infty$.

According to the Girsanov's theorem, we have the following equation

$$dF(Y_0^t) = f_t d\widetilde{Y}_t + f_t l_t F(Y_0^t)dt. \tag{43}$$

The solution of the SDE (43) is

$$F(Y_0^t) = u_t \int_0^t \frac{f_s}{u_s} d\widetilde{Y}_s. \tag{44}$$

Substituting (44) into (42), we obtain

$$\begin{aligned} L(\lambda) &= \Big(\frac{u_T}{u_0}\Big)^{\frac{1}{2}} \widetilde{E}\Big[\Psi\Big(\int_0^T g_t dY_t + \int_0^T q_t f_t dY_t\Big)\Big] \\ &= \Big(\frac{u_T}{u_0}\Big)^{\frac{1}{2}} \widetilde{E}\Big[\Psi\Big(\int_0^T g_t d\widetilde{Y}_t + \int_0^T (g_t l_t + q_t)u_t\Big(\int_0^t \frac{f_s}{u_s} d\widetilde{Y}_s\Big)dt\Big)\Big] \\ &= \Big(\frac{u_T}{u_0}\Big)^{\frac{1}{2}} \widetilde{E}\Big[\Psi\Big(\int_0^T m_t d\widetilde{Y}_t\Big)\Big]. \end{aligned} \tag{45}$$

This proves result (33) or (34). We know that $\Psi(\cdot)$ is an exponential function. Since the function m_t is non-random, $\int_0^t m_s d\widetilde{Y}_s \sim N(0, m_t^2)$, we know $\widetilde{E}\Big[\Psi\Big(\int_0^T m_t d\widetilde{Y}_t\Big)\Big] = e^{\frac{1}{2}\lambda^2\rho^2}$, where $\Psi(x) = e^{-\lambda x}$.

5 A solution about the optimal problem

In this section we provide a basic method which can solve the optimization problem under the given utility function. Applying the representation of $\widehat{Z}_t$ and the Laplace transform of $\log(\widehat{Z}_T/C)$, we choose $U(x) = 1 - \frac{1}{\alpha}e^{-\alpha x}$. Therefore, we have

$$U'(x) = e^{-\alpha x}.$$

The inverse of U' is defined by

$$I(y) = \log y^{-\frac{1}{\alpha}}.$$

Thus,

$$U \circ I(z) = 1 - \frac{1}{\alpha} z. \tag{46}$$

Generally, by the Kalman filter theorem, the conditional density of $\widehat{Z}_T$ depends on the path up to the time t. $L_T(\lambda)$ is henceforth defined as

$$L_T(\lambda) = E\Big[e^{-\alpha \log(\widehat{Z}_T/C)} | \mathcal{F}_t\Big]. \tag{47}$$

When $t = 0$, we denote $L_0(\lambda) = L(\lambda)$.

Remark 3. If given time $t < T$, Y_0^t can be observed. However, Y_t^T can't be defined. For this reason, the representation of $\widehat{Z}_T$ is written as two parts, one of which depends on the past information, the other is not.

At time t, we define

$$\widehat{Z}_T = C z_0^t Z_t^T, \tag{48}$$

where

$$\begin{aligned} Z_0^t = {} & e^{AF^2(Y_0^t) + \int_0^t (p_s + (k+1)h_s) F^2(Y_0^t) ds + \int_t^T (p_s + (k+1)h_s) F^2(Y_0^s) ds + G(Y_0^t)} \\ & \times e^{\int_0^t q_s F(Y_0^s) ds + \int_t^T q_s F(Y_0^t) ds} \end{aligned} \tag{49}$$

and

$$\begin{aligned} Z_t^T = {} & e^{A\left(F^2(Y_0^T) + 2F(Y_0^t)F(Y_t^T)\right) + \int_t^T (p_s + (k+1)h_s)\left(2F(Y_0^t)F(Y_t^s) + F^2(Y_t^s)\right) ds} \\ & \times e^{G(Y_t^T) + \int_t^T q_s F(Y_t^s) ds}. \end{aligned} \tag{50}$$

It shows that $L(\lambda)$ can be observed on the interval $[0, t]$. By employing the conditional expectation formula, the $L_t(\lambda)$ is directly obtained.

$$L_t(\lambda) = (z_0^t)^{-\lambda} \left(\frac{u_T}{u_t}\right)^{\frac{1}{2}} e^{\frac{1}{2}\lambda^2 p_t^2}. \tag{51}$$

Before computing the value function and the wealth process, it is necessary to find $\tau(X_0)$. Using (11), we know

$$X_0 = \widetilde{E}\Big[\log\left(\tau(X_0)\widehat{Z}_T\right)^{-\frac{1}{\alpha}}\Big], \tag{52}$$

from which we have

$$e^{X_0} = (C\tau(X_0)))^{-\frac{1}{\alpha}}\widetilde{E}\Big[e^{-\frac{1}{\alpha}\log(\widehat{Z}_T/C)}\Big] = (C\tau(X_0))L\Big(\frac{1}{\alpha}\Big),$$

which derives

$$\tau(X_0) = \frac{1}{C}\Big(\frac{e^{X_0}}{L(1/\alpha)}\Big)^{-\alpha} = \frac{1}{C}e^{-\alpha X_0}L(1). \tag{53}$$

Substituting (46) and (53) into (12), the value function is turned into

$$\begin{aligned} v(X_0) &= 1 - \frac{1}{\alpha}E\Big[\Big(\frac{e^{X_0}}{L(1/\alpha)}\Big)^{-\alpha}\frac{\widehat{Z}_T}{C}\Big] = 1 - \frac{1}{\alpha}\widetilde{E}\Big[\Big(\frac{e^{X_0}}{L(1/\alpha)}\Big)^{-\alpha}\frac{1}{C}\Big] \\ &= 1 - \frac{1}{\alpha C}e^{-\alpha X_0}L(1) = 1 - \frac{1}{\alpha C}e^{-\alpha X_0}\Big(\frac{u_T}{u_t}\Big)^{\frac{1}{2}}e^{\frac{1}{2}\rho^2}, \end{aligned} \tag{54}$$

where ρ is defined by (37).

According to Theorem 4.1 and using the model (13), we write the wealth process

$$\begin{aligned} X_t &= \widetilde{E}\Big[I(\tau(X_0)\widehat{Z}_T)|\mathcal{F}_t\Big] = \widetilde{E}\Big[\log\left(\tau(X_0)\widehat{Z}_T\right)^{-\frac{1}{\alpha}}|\mathcal{F}_t\Big] \\ &= \widetilde{E}\Big[\log\Big(\Big(\frac{e^{X_0}}{L(1/\alpha)}\Big)^{-\alpha}\frac{\widehat{Z}_T}{C}\Big)^{-\frac{1}{\alpha}}|\mathcal{F}_t\Big] \\ &= \log\Big(\frac{e^{X_0}}{L(1/\alpha)}\Big)\widetilde{E}\Big[\log e^{-\frac{1}{\alpha}\log(\widehat{Z}_T/C)}|\mathcal{F}_t\Big] \\ &= X_0\log\Big(\frac{L_t(1/\alpha)}{L(1/\alpha)}\Big), \end{aligned} \tag{55}$$

where the time $t < T$.

Substituting (51) into (55), the wealth process X_t can be written as

$$X_t = X_0\frac{\rho_t^2 - \rho_0^2}{2\alpha^2}\log\Big((Z_0^t)^{-\frac{1}{\alpha}}\Big(\frac{u_0}{u_t}\Big)^{\frac{1}{2}}\Big), \tag{56}$$

where ρ is defined by (37) and Z_0^t is defined by (49).

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