

Mathematica Aeterna, Vol. 1, 2011, no. 07, 461- 490

On the construction of N-dimensional hypernumbers

John F. Moxnes

Department for Protection

Norwegian Defence Research Establishment

P.O. Box 25

2007 Kjeller, Norway

E-mail: john-f.moxnes@ffi.no

Tel.: +47 63 807514, Fax: +47 63 807509

Kjell Hausken

Faculty of Social Sciences

University of Stavanger

4036 Stavanger, Norway

E-mail: kjell.hausken@uis.no

Tel.: +47 51 831632, Tel. Dept: +47 51 831500, Fax: +47 51 831550

Abstract

Complex numbers extend the concept of the 1 dimensional numbers to 2 dimensions. Quaternions extend numbers to 4 dimensions. Octonions and sedenions are extensions to 8 and 16 dimensions respectively. We study a general form of complex numbers, various axiomatic structures of 3 dimensional numbers, and finally N dimensional numbers, $N = 2^k$, $k=0,1,2,\dots$. Quaternions, octonions and sedenions are special cases. Two different structures for addition are studied.

Keywords: complex numbers, hypernumbers, quaternions, octonions, sedenions

1 Introduction

1.1 Background

Man invented natural numbers to count people, sheep, cattle, etc. Fractions (rational numbers) were probably invented to account for fractions of volumes or areas, e.g. of corn, water or land. Probably zero was invented together with the construction of a number positioning system analogous to the abacus. Negative numbers account for dept (something owed to someone else). The irrational numbers were noticed by the Greeks. The numbers were necessary to account for the length of all geometric objects defined by Euclidean geometry. Complex numbers were historically introduced to allow for solutions of certain equations that have no real number solution, i.e. $x^2=-1$ (Nahin 1998). Complex numbers extend the concept of the 1 dimensional numbers to 2 dimensions. When complex numbers are viewed as a position vector in a 2 dimensional Cartesian system (complex plane), the x-axis is used for the real part and the vertical axis for the imaginary part. In this way a subspace of the complex numbers arises, i.e. the numbers on the x-axis are isomorphic to the real numbers. Addition of complex numbers corresponds to well known vector addition, while complex multiplication corresponds to multiplying magnitudes of the two vectors and adding their rotational angle with the x-axis. The commutative and associative laws for multiplication (and addition) are fulfilled for the complex numbers. Complex numbers are today used in many scientific fields such as engineering, electromagnetism, quantum physics and applied mathematics. The addition rule of complex numbers was constructed before vector addition and was most likely of vital importance for the idea of vector addition. A second type of complex numbers is the so called split-complex numbers. The square of the imaginary number i is 1 and not -1. These numbers were introduced by Cockle (1848) and Clifford (1882). For the third well known type of complex numbers, i.e. dual numbers, the square of the imaginary number is $(0,0)$.

Nonstandard numbers are often referred to as hypernumbers. Along with the imaginary hypernumber i another hypernumber emerged within physics in the nineteenth century (Cayley 1845, Clifford 1873, Hamilton 1969). Quaternions are numbers of 4 dimensions where the base elements are real numbers. Historically quaternions were visualized as the quotient of two directed lines in a 3 dimensional space. They are associative but not commutative in multiplication. Quaternions are now used in both theoretical and applied mathematics and in particular for calculations involving 3 dimensional rotations such as in 3 dimensional computer graphics and computer vision (Evans 1977, Conway et al. 2003). It is also common for the attitude control system of spacecraft to be commanded in terms of quaternions. Quaternions can be defined as two pairs of complex numbers with a certain

multiplication rule for two such pairs. The 3 dimensional imaginary space corresponds to the three dimensional vector space of elementary vector calculus.

Octonions are numbers of 8 dimensions. The octonions were developed by Graves (1843), inspired by the work of his friend Hamilton on quaternions. They were developed independently by Cayley (1845). Analogously to quaternions, the octonions can be defined as one pair of quaternions numbers with a certain multiplication rule for the pairs (Baez 2002). Octonions have some interesting properties related to Lie groups. Octonions have applications in string theory, special relativity and quantum logic. To day the concept of bi-quaternions are well known for pairs of quaternions. In general the elements in the bi-quaternions can be complex, dual or split-complex numbers. All these numbers can be conceived as 8 dimensional numbers. The space of bi-quaternions has a topology in the Euclidean metric on 8 dimensional space. The concepts of special relativity are illustrated through the bi-quaternions (Girard 1984).

Sedenions are extensions to 16 dimensions (Muses 1976, 1980, 1994, Carmondy 1988, Carmondy 1997). The geometry of these numbers has been exploited (Carmondy 1997).

Hypernumbers can be used to simplify the algebra of real numbers or extend established algebraic operations. In the literature the complex algebra extended to 4, 8 or 16 dimensional algebras are motivated mathematically by various assumptions about the norm and multiplication rules. No one within the literature has chosen a more generic route which is our objective in this paper. We do not explore the numbers' geometric aspects and their application to solve or simplify physical problems which are topics of future research.

1.2 This paper's contribution

Earlier research has confined attention to 1,2,4,8,16 dimensional numbers and the Clifford algebra. We are not aware of research on higher dimensional numbers than 16, and research not assuming the Clifford algebra. To provide a more generic foundation, we study a general form of complex numbers, various axiomatic structures of 3 dimensional numbers, and finally N dimensional numbers, $N = 2^k$, $k=0,1,2,\dots$, based on the general form of complex numbers. We proceed outside the Clifford algebra in sections 7 and 8. Our approach is different from the literature in the sense that a new form of addition different from vector addition, exemplified with $(x_1, x_2) + (y_1, y_2) = (x_1 y_2 + y_1 x_2, x_2 y_2)$, is studied and generalized to $N = 2^k$, $k=0,1,2,\dots$

1.3 Organization

In section 2 we give an introduction to well known relations of the complex numbers and also explore some numerical results in two dimensions. In section 3 we study 3 dimensional numbers. In section 4 we study 4 dimensional numbers and quaternions. In section 5 we study 8 dimensional numbers and octonions in particular. In section 6 we study N dimensional numbers that add as vectors. In section 7 we study quite generally numbers that do not add as vectors. Such numbers have not been studied in the literature. Section 8 provides our most general structure of N dimensional numbers. Section 9 concludes.

2 Complex numbers and some extensions as an introduction

A complex number is a special type of an ordered pair. We define an ordered pair by $(x_1, x_2) \stackrel{def}{=} \{x_1, \{x_1, x_2\}\}$, $x_1, x_2 \in R$. “def” means definition. The axiomatic structure of the complex numbers is

$$\begin{aligned} (x_1, x_2) + (y_1, y_2) &\stackrel{mod}{=} (x_1 + y_1, x_2 + y_2), (a) \\ (x_1, x_2) \times (y_1, y_2) &\stackrel{mod}{=} (x_1 y_1 - x_2 y_2, x_1 y_2 + x_2 y_1), (b) \end{aligned} \tag{2.1}$$

where “mod” means model assumption. It follows that

$$\begin{aligned} (x_1, 0) \times (y_1, y_2) &= (x_1 y_1, x_1 y_2) = (y_1, y_2) \times (x_1, 0) \\ (x, 0) \times (y_1, y_2) &= (x y_1, x y_2) = \underbrace{(y_1, y_2)}_{x \text{ times}} + \dots, x \in N \\ (0, x_2) \times (0, y_2) &= (x_2 y_2, 0) \times ((0, 1) \times (0, 1)) \\ (x_1, x_2) \times (y_1, y_2) &= (y_1, y_2) \times (x_1, x_2) \\ (1, 0) \times (1, 0) &= (1, 0) \\ (0, 1) \times (0, 1) &= (-1, 0) \end{aligned} \tag{2.2}$$

The distributive law is fulfilled:

$$\begin{aligned} ((x_1, x_2) + (y_1, y_2)) \times (z_1, z_2) &= (x_1 + y_1, x_2 + y_2) \times (z_1, z_2) \\ &= ((x_1 + y_1)z_1 - (x_2 + y_2)z_2, (x_1 + y_1)z_2 + (x_2 + y_2)z_1) \\ &= (x_1 z_1 - x_2 z_2, x_1 z_2 + x_2 z_1) + (y_1 z_1 - y_2 z_2, y_1 z_2 + y_2 z_1) = (x_1, x_2) \times (z_1, z_2) + (y_1, y_2) \times (z_1, z_2) \end{aligned} \tag{2.3}$$

