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A Unique Common 3-tupled fixed point theorem for $\psi - \phi$ contractions in partial metric spaces

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Abstract

In this paper, we obtain a unique common 3-tupled fixed point theorem in partial metric spaces and also mention an example to support our theorem .

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1 Introduction

The notion of partial metric space was introduced by Matthews [17] as a part of the study of denotational semantics of data flow networks. In fact, it is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation and domain theory in computer science (see e.g. [23, 26, 27, 8, 14, 20, 15, 21, 24]).

Matthews [17, 18], Oltra and Valero [19] and Romaguera [22] and Altun et al. [2] proved some fixed point theorems in partial metric spaces for a single map (see also [1, 9, 10, 11, 12, 13, 3, 4, 5, 25, 20]).

Recently, the concept of coupled fixed points was introduced by Bhaskar and Lakshmikantham [7]. In [4], Aydi proved some coupled fixed point theorems for the mappings satisfying contractive conditions in partial metric spaces.

In this paper, we introduce a 3-tupled fixed point and obtain a unique common 3-tupled fixed point theorem for four self mappings satisfying a $\psi-\phi$ contractive condition in partial metric spaces.

First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

2 Preliminary Notes

Definition 2.1 *(See [17, 18]) A partial metric on a nonempty set X is a function $p : X \times X \to R^+$ such that for all $x, y, z \in X$:*

(p_1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p_2) $p(x,x) \leq p(x,y), p(y,y) \leq p(x,y)$,

(p_3) $p(x,y) = p(y,x)$,

(p_4) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair (X,p) is called a partial metric space (PMS).

If p is a partial metric on X, then the function $p^s : X \times X \to R^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{1}$$

is a metric on X.

Example 2.2 *(See e.g. [18, 10, 11, 1]) Consider $X = [0,\infty)$ with $p(x,y) = \max\{x,y\}$. Then (X,p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $p^s(x,y) = |x-y|$.*

Example 2.3 *(See [9]) Let $X = \{[a,b] : a,b,\in R,\ a \leq b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric space.*

Example 2.4 *(See [9]) Let* $X := [0,1]\cup[2,3]$ *and define* $p : X\times X \to [0,\infty)$ *by*

$$p(x,y) = \begin{cases} \max\{x,y\} & \text{if } \{x,y\}\cap[2,3]\neq\emptyset, \\ |x-y| & \text{if } \{x,y\}\subset[0,1]. \end{cases}$$

Then (X, p) is a complete partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\varepsilon),\ x\in X, \varepsilon>0\}$, where $B_p(x,\varepsilon) = \{y\in X : p(x,y) < p(x,x)+\varepsilon\}$ for all $x\in X$ and $\varepsilon>0$.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (see e.g. [17, 18, 2, 1, 10, 13].)

Definition 2.5

1. *A sequence* $\{x_n\}$ *in the PMS* (X,p) *converges to the limit* x *if and only if* $p(x,x) = \lim\limits_{n\to\infty} p(x,x_n)$.
2. *A sequence* $\{x_n\}$ *in the PMS* (X,p) *is called a Cauchy sequence if* $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ *exists and is finite.*
3. *A PMS* (X,p) *is called complete if every Cauchy sequence* $\{x_n\}$ *in* X *converges with respect to* τ_p*, to a point* $x\in X$ *such that* $p(x,x) = \lim\limits_{n,m\to\infty} p(x_n,x_m)$.
4. *A mapping* $F : X\to X$ *is said to be continuous at* $x_0\in X$ *if for every* $\epsilon>0$*, there exists* $\delta>0$ *such that* $F(B_p(x_0,\delta))\subseteq B_p(Fx_0,\epsilon)$.

The following lemma is one of the basic results in PMS([17, 18, 1, 2, 10, 13]).

Lemma 2.6

1. *A sequence* $\{x_n\}$ *is a Cauchy sequence in the PMS* (X,p) *if and only if it is a Cauchy sequence in the metric space* (X,p^s).
2. *A PMS* (X,p) *is complete if and only if the metric space* (X,p^s) *is complete. Moreover*

$$\lim_{n\to\infty} p^s(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m) \qquad (2)$$

Next, we give two lemmas which will be used in the proof of our main result. For the proofs we refer to e.g. [1, 10, 11].

Lemma 2.7 *Assume* $x_n\to z$ *as* $n\to\infty$ *in a PMS* (X,p) *such that* $p(z,z)=0$*. Then* $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ *for every* $y\in X$.

Lemma 2.8 *Let* (X,p) *be a PMS. Then*

(A) If $p(x,y)=0$ *then* $x=y$,

(B) If $x\neq y$, *then* $p(x,y)>0$,

(C) If $x=y$, $p(x,y)$ *may not be* 0.

Lemma 2.9 *If* $\{x_n\}$ *is converges to* x *in* (X,p), *then* $\lim_{n\to\infty} p(x_n,y)\leq p(x,y)$ *for all* $y\in X$.

Definition 2.10 *[7]. An element* $(x,y)\in X\times X$ *is called a coupled fixed point of mapping* $F:X\times X\to X$ *if* $x=F(x,y)$ *and* $y=F(y,x)$.

Very recently Berinde and Borcut [6] introduced the notion of tripled fixed point of a mapping as follows

Definition 2.11 *[6]. An element* $(x,y,z)\in X\times X\times X$ *is called a tripled fixed point of mapping* $F:X\times X\times X\to X$ *if* $x=F(x,y,z)$, $y=F(y,x,y)$ *and* $z=F(z,y,x)$.

Based on these difinitions, we give the following definitions.

Definition 2.12 *An element* $(x,y,z)\in X\times X\times X$ *is called a 3-tupled fixed point of the mapping* $T:X\times X\times X\to X$ *if* $x=T(x,y,z), y=T(y,z,x)$ *and* $z=T(z,y,x)$.

Definition 2.13 *An element* $(x,y,z)\in X\times X\times X$ *is called*
(i) *a 3-tupled coincident point of mappings* $T:X\times X\times X\to X$ *and* $f:X\to X$ *if* $fx=T(x,y,z)$, $fy=T(y,z,x)$ *and* $fz=T(z,x,y)$;
(ii) *a common 3-tupled fixed point of mappings* $T:X\times X\times X\to X$ *and* $f:X\to X$ *if* $x=fx=T(x,y,z)$, $y=fy=T(y,z,x)$ *and* $z=fz=T(z,x,y)$.

Definition 2.14 *The mappings* $T:X\times X\times X\to X$ *and* $f:X\to X$ *are called* w *- compatible if* $f(T(x,y,z))=T(fx,fy,fz)$, $f(T(y,z,x))=T(fy,fz,fx)$ *and* $f(T(z,x,y))=T(fz,fx,fy)$ *whenever* $fx=T(x,y,z)$, $fy=T(y,z,x)$ *and* $fz=T(z,x,y)$.

Now we prove our main result.

3 Main Results

Let Ψ denotes the set of all functions $\psi : [0,\infty) \to [0,\infty)$ satisfying
(ψ_1) ψ is continuous and non-decreasing,
(ψ_2) $\psi(t) = 0$ if and only if $t = 0$,
(ψ_3) $\psi(t+s) \leq \psi(t) + \psi(s)$, for all $t, s \in [0,\infty)$,
while Φ denotes the set of all functions $\phi : [0,\infty) \to [0,\infty)$ satisfying
(ϕ_1) ϕ is continuous,
(ϕ_2) $\phi(t) = 0$ if and only if $t = 0$.

Theorem 3.1 *Let (X,p) be a partial metric space and let $S, T : X \times X \times X \to X$ and $f, g : X \to X$ be mappings satisfying*

$$(i)\quad \psi(p(S(x,y,z),T(u,v,w))) \leq \frac{1}{3}\psi(p(fx,gu)+p(fy,gv)+p(fz,gw)) - \phi(p(fx,gu)+p(fy,gv)+p(fz,gw))$$

$\forall x,y,z,u,v,w \in X$ where $\psi \in \Psi$ and $\phi \in \Phi$,
(ii) $S(X \times X \times X) \subseteq g(X), T(X \times X \times X) \subseteq f(X)$,
(iii) either $f(X)$ or $g(X)$ is a complete subspace of X,
(iv) the pairs (f,S) and (g,T) are w - compatible.
Then S, T, f and g have a unique common 3-tupled fixed point in $X \times X \times X$ of the form (α,α,α).

Proof 3.2 *Let x_0, y_0, z_0 be arbitrary points in X. From (ii), there exist sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ in X such that*

$$\begin{aligned} u_{2n} &= gx_{2n+1} = S(x_{2n},y_{2n},z_{2n}),\\ v_{2n} &= gy_{2n+1} = S(y_{2n},z_{2n},x_{2n}),\\ w_{2n} &= gz_{2n+1} = S(z_{2n},x_{2n},y_{2n}),\\ u_{2n+1} &= fx_{2n+2} = T(x_{2n+1},y_{2n+1},z_{2n+1}),\\ v_{2n+1} &= fy_{2n+2} = T(y_{2n+1},z_{2n+1},x_{2n+1}),\\ w_{2n+1} &= fz_{2n+2} = T(z_{2n+1},x_{2n+1},y_{2n+1}), \qquad n = 0,1,2,\cdots \end{aligned} \tag{3}$$

Then by (i), we have

$$\begin{aligned} \psi(p(u_2,u_1)) &= \psi(p(S(x_2,y_2,z_2),T(x_1,y_1,z_1)))\\ &\leq \frac{1}{3}\psi(p(fx_2,gx_1)+p(fy_2,gy_1)+p(fz_2,gz_1))\\ &\quad -\phi(p(fx_2,gx_1)+p(fy_2,gy_1)+p(fz_2,gz_1))\\ &= \frac{1}{3}\psi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0))\\ &\quad -\phi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0)). \end{aligned}$$

$$\begin{aligned} \psi(p(v_2,v_1)) &= \psi(p(S(y_2,z_2,x_2),T(y_1,z_1,x_1)))\\ &\leq \frac{1}{3}\psi(p(fy_2,gy_1)+p(fz_2,gz_1)+p(fx_2,gx_1))\\ &\quad -\phi(p(fy_2,gy_1)+p(fz_2,gz_1)+p(fx_2,gx_1))\\ &= \frac{1}{3}\psi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0))\\ &\quad -\phi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0)). \end{aligned}$$

And

$$\begin{aligned}\psi(p(w_2,w_1)) &= \psi(p(S(z_2,x_2,y_2),T(z_1,x_1,y_1))) \\ &\le \tfrac{1}{3}\psi(p(fz_2,gz_1)+p(fx_2,gx_1)+p(fy_2,gy_1)) \\ &\qquad -\phi(p(fz_2,gz_1)+p(fx_2,gx_1)+p(fy_2,gy_1)) \\ &= \tfrac{1}{3}\psi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0)) \\ &\qquad -\phi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0)).\end{aligned}$$

By definition of ψ, we have

$$\begin{aligned}\psi(p(u_2,u_1)+p(v_2,v_1)+p(w_2,w_1)) &\le \psi(p(u_2,u_1))+\psi(p(v_2,v_1))+\psi(p(w_2,w_1)) \\ &\le \psi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0)) \\ &\qquad -3\phi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0)) \\ &\le \psi(p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0)).\end{aligned}$$

Since ψ is non - decreasing, we have

$$p(u_2,u_1)+p(v_2,v_1)+p(w_2,w_1) \le p(u_1,u_0)+p(v_1,v_0)+p(w_1,w_0).$$

Similarly

$$\begin{aligned}\psi(p(u_3,u_2)+p(v_3,v_2)+p(w_3,w_2)) &\le \psi(p(u_2,u_1)+p(v_2,v_1)+p(w_2,w_1)) \\ &\qquad -3\phi(p(u_2,u_1)+p(v_2,v_1)+p(w_2,w_1)).\end{aligned}$$

Hence by induction, we have

$$\begin{aligned}\psi\begin{pmatrix} p(u_n,u_{n+1}) \\ +p(v_n,v_{n+1}) \\ +p(w_n,w_{n+1})\end{pmatrix} &\le \psi\left(p(u_n,u_{n-1})+p(v_n,v_{n-1})+p(w_n,w_{n-1})\right) \\ &\quad -3\phi\left(p(u_n,u_{n-1})+p(v_n,v_{n-1})+p(w_n,w_{n-1})\right) \quad (4) \\ &\le \psi(p(u_n,u_{n-1})+p(v_n,v_{n-1})+p(w_n,w_{n-1})).\end{aligned}$$

Since ψ is non - decreasing, we have

$$p(u_n,u_{n+1})+p(v_n,v_{n+1})+p(w_n,w_{n+1}) \le p(u_n,u_{n-1})+p(v_n,v_{n-1})+p(w_n,w_{n-1}).$$

Put $R_n = p(u_n,u_{n+1})+p(v_n,v_{n+1})+p(w_n,w_{n+1})$. Then $\{R_n\}$ is a non - increasing sequence of real numbers and must converge to $r \ge 0$.
Suppose $r > 0$.
Letting $n \to \infty$ in (4), we have that

$$\begin{aligned}\psi(r) &\le \psi(r)-\phi(r) \\ &< \psi(r).\end{aligned}$$

It is a contradiction. Hence $r = 0$.
Thus

$$\lim_{n\to\infty}[p(u_n,u_{n+1})+p(v_n,v_{n+1})+p(w_n,w_{n+1})] = 0.$$

It follows that

$$\lim_{n\to\infty} p(u_n, u_{n+1}) = \lim_{n\to\infty} p(v_n, v_{n+1}) = \lim_{n\to\infty} p(w_n, w_{n+1}) = 0. \tag{5}$$

From (p_2), *we have*

$$\lim_{n\to\infty} p(u_n, u_n) = \lim_{n\to\infty} p(v_n, v_n) = \lim_{n\to\infty} p(w_n, w_n) = 0. \tag{6}$$

From definition of p^s, *(2.3) and (2.4), we have*

$$\lim_{n\to\infty} p^s(u_n, u_{n+1}) = 0. \tag{7}$$

Similarly

$$\lim_{n\to\infty} p^s(v_n, v_{n+1}) = 0 \tag{8}$$

and

$$\lim_{n\to\infty} p^s(w_n, w_{n+1}) = 0. \tag{9}$$

Now we prove that $\{u_{2n}\}, \{v_{2n}\}$ *and* $\{w_{2n}\}$ *are Cauchy sequences.*
On contrary, suppose that $\{u_{2n}\}$ *or* $\{v_{2n}\}$ *or* $\{w_{2n}\}$ *is not Cauchy.*
Then there exists an $\epsilon > 0$ *and monotone increasing sequences of natural numbers* $\{2m_k\}$ *and* $\{2n_k\}$ *such that* $n_k > m_k$,

$$p^s(u_{2m_k}, u_{2n_k}) + p^s(v_{2m_k}, v_{2n_k}) + p^s(w_{2m_k}, w_{2n_k}) \geq \epsilon \tag{10}$$

and

$$p^s(u_{2m_k}, u_{2n_k-2}) + p^s(v_{2m_k}, v_{2n_k-2}) + p^s(w_{2m}, w_{2n_k-2}) < \epsilon. \tag{11}$$

From (10) and (11), we have

$$\begin{aligned}
\epsilon &\leq p^s(u_{2m_k}, u_{2n_k}) + p^s(v_{2m_k}, v_{2n_k}) + p^s(w_{2m_k}, w_{2n_k}) \\
&\leq p^s(u_{2m_k}, u_{2n_k-2}) + p^s(v_{2m_k}, v_{2n_k-2}) + p^s(w_{2m_k}, w_{2n_k-2}) \\
&\quad + p^s(u_{2n_k-2}, u_{2n_k-1}) + p^s(v_{2n_k-2}, v_{2n_k-1}) + p^s(w_{2n_k-2}, w_{2n_k-1}) \\
&\quad + p^s(u_{2n_k-1}, u_{2n_k}) + p^s(v_{2n_k-1}, v_{2n_k}) + p^s(w_{2n_k-1}, w_{2n_k}) \\
&< \epsilon + p^s(u_{2n_k-2}, u_{2n_k-1}) + p^s(v_{2n_k-2}, v_{2n_k-1}) + p^s(w_{2n_k-2}, w_{2n_k-1}) \\
&\quad + p^s(u_{2n_k-1}, u_{2n_k}) + p^s(v_{2n_k-1}, v_{2n_k}) + p^s(w_{2n_k-1}, w_{2n_k}).
\end{aligned}$$

Letting $k \to \infty$ *and using (7), (8) and (9) we have*

$$\lim_{k\to\infty} [p^s(u_{2m_k}, u_{2n_k}) + p^s(v_{2m_k}, v_{2n_k}) + p^s(w_{2m_k}, w_{2n_k})] = \epsilon. \tag{12}$$

By definition of p^s *and (6), we have*

$$\lim_{k\to\infty} [p(u_{2m_k}, u_{2n_k}) + p(v_{2m_k}, v_{2n_k}) + p(w_{2m_k}, w_{2n_k})] = \frac{\epsilon}{2}. \tag{13}$$

Also,

$$\begin{aligned}\epsilon &\leq p^s(u_{2m_k}, u_{2n_k}) + p^s(v_{2m_k}, v_{2n_k}) + p^s(w_{2m_k}, w_{2n_k}) \\ &\leq p^s(u_{2m_k}, u_{2m_k-1}) + p^s(v_{2m_k}, v_{2m_k-1}) + p^s(w_{2m_k}, w_{2m_k-1}) \\ &\quad + p^s(u_{2m_k-1}, u_{2n_k}) + p^s(v_{2m_k-1}, v_{2n_k}) + p^s(w_{2m_k-1}, w_{2n_k}) \\ &\leq 2[p^s(u_{2m_k}, u_{2m_k-1}) + p^s(v_{2m_k}, v_{2m_k-1}) + p^s(w_{2m_k}, w_{2m_k-1})] \\ &\quad + [p^s(u_{2m_k}, u_{2n_k}) + p^s(v_{2m_k}, v_{2n_k}) + p^s(w_{2m_k}, w_{2n_k})].\end{aligned} \tag{14}$$

Letting $k \to \infty$ and using (7),(8), (9), (12) and (14), we have

$$\lim_{k\to\infty}[p^s(u_{2m_k-1}, u_{2n_k}) + p^s(v_{2m_k-1}, v_{2n_k}) + p^s(w_{2m_k-1}, w_{2n_k})] = \epsilon. \tag{15}$$

By definition of p^s and (6), we have

$$\lim_{k\to\infty}[p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})] = \frac{\epsilon}{2}. \tag{16}$$

On other hand we have

$$\begin{aligned} &p^s(u_{2m_k}, u_{2n_k}) + p^s(v_{2m_k}, v_{2n_k}) + p^s(w_{2m_k}, w_{2n_k}) \\ &\quad \leq p^s(u_{2m_k}, u_{2n_k+1}) + p^s(v_{2m_k}, v_{2n_k+1}) + p^s(w_{2m_k}, w_{2n_k+1}) \\ &\qquad + p^s(u_{2n_k+1}, u_{2n_k}) + p^s(v_{2n_k+1}, v_{2n_k}) + p^s(w_{2n_k+1}, w_{2n_k}).\end{aligned}$$

Letting $k \to \infty$ and using (6), (7),(8)(9) and (12), we have

$$\begin{aligned}\epsilon &\leq \lim_{k\to\infty}[p^s(u_{2m_k}, u_{2n_k+1}) + p^s(v_{2m_k}, v_{2n_k+1}) + p^s(w_{2m_k}, w_{2n_k+1})] + 0 \\ &= 2\lim_{k\to\infty}[p(u_{2m_k}, u_{2n_k+1}) + p(v_{2m_k}, v_{2n_k+1}) + p(w_{2m_k}, w_{2n_k+1})].\end{aligned}$$

Thus,

$$\frac{\epsilon}{2} \leq \lim_{k\to\infty}[p(u_{2m_k}, u_{2n_k+1}) + p(v_{2m_k}, v_{2n_k+1}) + p(w_{2m_k}, w_{2n_k+1})].$$

By the properties of ψ, we have

$$\begin{aligned}\psi\left(\frac{\epsilon}{2}\right) &\leq \psi\left(\lim_{k\to\infty}[p(u_{2m_k}, u_{2n_k+1}) + p(v_{2m_k}, v_{2n_k+1}) + p(w_{2m_k}, w_{2n_k+1})]\right) \\ &\leq \lim_{k\to\infty}\left(\begin{array}{c}\psi(p(u_{2m_k}, u_{2n_k+1})) + \psi(p(v_{2m_k}, v_{2n_k+1})) \\ +\psi(p(w_{2m_k}, w_{2n_k+1}))\end{array}\right).\end{aligned} \tag{17}$$

Now

$$\begin{aligned}\psi(p(u_{2m_k}, u_{2n_k+1})) &= \psi\left(p(S(x_{2m_k}, y_{2m_k}, z_{2m_k}), T(x_{2n_k+1}, y_{2n_k+1}, z_{2n_k+1}))\right) \\ &\leq \tfrac{1}{3}\psi\left(p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})\right) \\ &\quad -\phi\left(p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})\right).\end{aligned}$$

Similarly

$$\begin{aligned}\psi(p(v_{2m_k}, v_{2n_k+1})) &\leq \tfrac{1}{3}\psi\left(p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})\right) \\ &\quad -\phi\left(p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})\right)\end{aligned}$$

and

$$\begin{aligned}\psi(p(w_{2m_k}, w_{2n_k+1})) &\leq \tfrac{1}{3}\psi\left(p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})\right)\\ &\quad -\phi\left(p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})\right).\end{aligned}$$

Hence from (16) and (17), we have

$$\begin{aligned}\psi\left(\tfrac{\epsilon}{2}\right) &\leq \lim_{k\to\infty}\begin{pmatrix}\psi\left(p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})\right)\\ -3\phi\left(p(u_{2m_k-1}, u_{2n_k}) + p(v_{2m_k-1}, v_{2n_k}) + p(w_{2m_k-1}, w_{2n_k})\right)\end{pmatrix}\\ &= \psi\left(\tfrac{\epsilon}{2}\right) - 3\phi\left(\tfrac{\epsilon}{2}\right)\\ &< \psi\left(\tfrac{\epsilon}{2}\right).\end{aligned}$$

It is a contradiction.
Hence $\{u_{2n}\}, \{v_{2n}\}$ *and* $\{w_{2n}\}$ *are Cauchy sequences in the metric space* (X, p^s).
Letting $n, m \to \infty$ *in*
$|p^s(u_{2n+1}, u_{2m+1}) - p^s(u_{2n}, u_{2m})| \leq p^s(u_{2n+1}, u_{2n}) + p^s(u_{2m+1}, u_{2m})$
we get $\lim_{n,m\to\infty} p^s(u_{2n+1}, u_{2m+1}) = 0.$
Letting $n, m \to \infty$ *in*
$|p^s(v_{2n+1}, v_{2m+1}) - p^s(v_{2n}, v_{2m})| \leq p^s(v_{2n+1}, v_{2n}) + p^s(v_{2m+1}, v_{2m})$
we get $\lim_{n,m\to\infty} p^s(v_{2n+1}, v_{2m+1}) = 0.$
Letting $n, m \to \infty$ *in*
$|p^s(w_{2n+1}, w_{2m+1}) - p^s(w_{2n}, w_{2m})| \leq p^s(w_{2n+1}, w_{2n}) + p^s(w_{2m+1}, w_{2m})$
we get $\lim_{n,m\to\infty} p^s(w_{2n+1}, w_{2m+1}) = 0.$
Thus $\{u_{2n+1}\}, \{v_{2n+1}\}$ *and* $\{w_{2n+1}\}$ *are Cauchy sequences in the metric space* (X, p^s).
Hence $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ *are Cauchy sequences in the metric space* (X, p^s).
Hence we have that

$$\lim_{n,m\to\infty} p^s(u_m, u_n) = \lim_{n,m\to\infty} p^s(v_m, v_n) = \lim_{n,m\to\infty} p^s(w_m, w_n) = 0. \tag{18}$$

Now from definition of p^s *and from (6), we have*

$$\lim_{n,m\to\infty} p(u_m, u_n) = \lim_{n,m\to\infty} p(v_m, v_n) = \lim_{n,m\to\infty} p(w_m, w_n) = 0. \tag{19}$$

Suppose $f(X)$ *is complete.*
Since $\{u_{2n+1}\} \subseteq f(X), \{v_{2n+1}\} \subseteq f(X)$ *and* $\{w_{2n+1}\} \subseteq f(X)$ *are Cauchy sequences in the complete metric space* $(f(X), p^s)$, *it follows that the sequences* $\{u_{2n+1}\}, \{v_{2n+1}\}$ *and* $\{w_{2n+1}\}$ *are convergent in* $(f(X), p^s)$.
Thus $\lim_{n\to\infty} p^s(u_{2n+1}, \alpha) = 0$, $\lim_{n\to\infty} p^s(v_{2n+1}, \beta) = 0$ *and* $\lim_{n\to\infty} p^s(w_{2n+1}, \gamma) = 0$ *for some* α, β *and* γ *in* $f(X)$.
There exist $x, y, z \in X$ *such that* $\alpha = fx, \beta = fy$ *and* $\gamma = fz$.

Since $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ *are Cauchy sequences in* X *and* $\{u_{2n+1}\} \to \alpha$, $\{v_{2n+1}\} \to \beta$ *and* $\{w_{2n+1}\} \to \gamma$ *, it follows that* $\{u_{2n}\} \to \alpha$, $\{v_{2n}\} \to \beta$ *and* $\{w_{2n}\} \to \gamma$.
From Lemma 2.6(2), we have

$$p(\alpha,\alpha) = \lim_{n\to\infty} p(u_{2n},\alpha) = \lim_{n\to\infty} p(u_{2n+1},\alpha) = \lim_{n,\ m\to\infty} p(u_n,u_m). \tag{20}$$

$$p(\beta,\beta) = \lim_{n\to\infty} p(v_{2n},\beta) = \lim_{n\to\infty} p(v_{2n+1},\beta) = \lim_{n,\ m\to\infty} p(v_n,v_m) \tag{21}$$

and

$$p(\gamma,\gamma) = \lim_{n\to\infty} p(w_{2n},\gamma) = \lim_{n\to\infty} p(w_{2n+1},\gamma) = \lim_{n,\ m\to\infty} p(w_n,w_m). \tag{22}$$

From (19), (20), (21) and (22) we have

$$p(\alpha,\alpha) = \lim_{n\to\infty} p(u_{2n},\alpha) = \lim_{n\to\infty} p(u_{2n+1},\alpha) = 0. \tag{23}$$

$$p(\beta,\beta) = \lim_{n\to\infty} p(v_{2n},\beta) = \lim_{n\to\infty} p(v_{2n+1},\beta) = 0 \tag{24}$$

and

$$p(\gamma,\gamma) = \lim_{n\to\infty} p(w_{2n},\gamma) = \lim_{n\to\infty} p(w_{2n+1},\gamma) = 0. \tag{25}$$

Now

$$\begin{aligned} p(S(x,y,z),\alpha) &\leq p(S(x,y,z),u_{2n+1}) + p(u_{2n+1},\alpha) - p(u_{2n+1},u_{2n+1}) \\ &\leq p(S(x,y,z),T(x_{2n+1},y_{2n+1},z_{2n+1})) + p(u_{2n+1},\alpha). \end{aligned}$$

Letting $n \to \infty$*, we have*

$$p(S(x,y,z),\alpha) \quad \leq \lim_{n\to\infty} p(S(x,y,z),T(x_{2n+1},y_{2n+1},z_{2n+1})) + 0.$$

Since by the property of ψ*, we have*

$$\begin{aligned} \psi(p(S(x,y,z),\alpha)) &\leq \lim_{n\to\infty} \psi(p(S(x,y,z),T(x_{2n+1},y_{2n+1},z_{2n+1})) \\ &\leq \lim_{n\to\infty} \left\{ \begin{array}{l} \frac{1}{3}\psi(p(fx,gx_{2n+1}) + p(fy,gy_{2n+1}) + p(fz,gz_{2n+1})) \\ -\phi(p(fx,gx_{2n+1}) + p(fy,gy_{2n+1}) + p(fz,gz_{2n+1})) \end{array} \right\} \\ &= \lim_{n\to\infty} \left\{ \begin{array}{l} \frac{1}{3}\psi(p(\alpha,u_{2n}) + p(\beta,v_{2n}) + p(\gamma,w_{2n})) \\ -\phi(p(\alpha,u_{2n}) + p(\beta,v_{2n}) + p(\gamma,w_{2n})) \end{array} \right\} \\ &= \psi(0) - \phi(0) = 0. \end{aligned}$$

It follows that $S(x,y,z) = \alpha$.
Similarly $S(y,z,x) = \beta$ *and* $S(z,x,y) = \gamma$.
Thus

$$\alpha = fx = S(x,y,z), \beta = fy = S(y,z,x) \text{ and } \gamma = fz = S(z,x,y).$$

Since (f, S) *is* w*-compatible we have*

$$S(\alpha,\beta,\gamma) = f\alpha,\ \ S(\beta,\gamma,\alpha) = f\beta \text{ and } S(\gamma,\alpha,\beta) = f\gamma.$$

By definition

$$p^s(f\alpha, u_{2n}) = 2p(f\alpha, u_{2n}) - p(f\alpha, f\alpha) - p(u_{2n}, u_{2n}).$$

Letting $n \to \infty$*, we get*

$$p^s(f\alpha, \alpha) = 2\lim_{n\to\infty} p(f\alpha, u_{2n}) - p(f\alpha, f\alpha) - 0.$$

Thus we have

$$2p(f\alpha,\alpha) - p(f\alpha, f\alpha) - p(\alpha,\alpha) = 2\lim_{n\to\infty} p(f\alpha, u_{2n}) - p(f\alpha, f\alpha) - 0.$$

Hence $\lim_{n\to\infty} p(f\alpha, u_{2n}) = p(f\alpha,\alpha)$*. By the same analogy, we have*

$$\lim_{n\to\infty} p(f\beta, v_{2n}) = p(f\beta,\beta) \text{ and } \lim_{n\to\infty} p(f\gamma, w_{2n}) = p(f\gamma,\gamma).$$

Now, we shall prove that $f\alpha = \alpha,\ f\beta = \beta$ *and* $f\gamma = \gamma$*.*
By the triangle inequality (p_4)*, we have*

$$\begin{aligned} p(f\alpha,\alpha) &\leq p(f\alpha, u_{2n+1}) + p(u_{2n+1},\alpha) - p(u_{2n+1}, u_{2n+1}) \\ &\leq p(S(\alpha,\beta,\gamma), T(x_{2n+1}, y_{2n+1}, z_{2n+1})) + p(u_{2n+1},\alpha). \end{aligned}$$

Letting $n \to \infty$ *in the equality above , we get*

$$p(f\alpha,\alpha) \ \ \leq \lim_{n\to\infty} p(S(\alpha,\beta,\gamma), T(x_{2n+1}, y_{2n+1}, z_{2n+1})) + 0.$$

Since ψ *is continuous and non - decreasing, we have*

$$\begin{aligned} \psi(p(f\alpha,\alpha)) &\leq \lim_{n\to\infty} \psi(p(S(\alpha,\beta,\gamma), T(x_{2n+1}, y_{2n+1}, z_{2n+1}))) \\ &\leq \lim_{n\to\infty} \begin{pmatrix} \frac{1}{3}\psi(p(f\alpha, gx_{2n+1}) + p(f\beta, gy_{2n+1}) + p(f\gamma, gz_{2n+1})) \\ -\phi(p(f\alpha, gx_{2n+1}) + p(f\beta, gy_{2n+1}) + p(f\gamma, gz_{2n+1})) \end{pmatrix} \\ &= \quad \lim_{n\to\infty} \begin{pmatrix} \frac{1}{3}\psi(p(f\alpha, u_{2n}) + p(f\beta, v_{2n}) + p(f\gamma, w_{2n})) \\ -\phi(p(f\alpha, u_{2n}) + p(f\beta, v_{2n}) + p(f\gamma, w_{2n})) \end{pmatrix} \\ &= \tfrac{1}{3}\psi(p(f\alpha,\alpha) + p(f\beta,\beta) + p(f\gamma,\gamma)) - \phi(p(f\alpha,\alpha) + p(f\beta,\beta) + p(f\gamma,\gamma)). \end{aligned}$$

Similarly

$$\psi(p(f\beta,\beta)) \ \ \leq \tfrac{1}{3}\psi(p(f\alpha,\alpha) + p(f\beta,\beta) + p(f\gamma,\gamma)) - \phi(p(f\alpha,\alpha) + p(f\beta,\beta) + p(f\gamma,\gamma))$$

and

$$\psi(p(f\gamma,\gamma)) \ \ \leq \tfrac{1}{3}\psi(p(f\alpha,\alpha) + p(f\beta,\beta) + p(f\gamma,\gamma)) - \phi(p(f\alpha,\alpha) + p(f\beta,\beta) + p(f\gamma,\gamma)).$$

By definition of ψ*, we have*

$$\begin{aligned}&\psi(p(f\alpha,\alpha)+p(f\beta,\beta)+p(f\gamma,\gamma))\\ &\quad\leq\psi(p(f\alpha,\alpha))+\psi(p(f\beta,\beta))+\psi(p(f\gamma,\gamma))\\ &\quad\leq\psi(p(f\alpha,\alpha)+p(f\beta,\beta)+p(f\gamma,\gamma))-3\phi(p(f\alpha,\alpha)+p(f\beta,\beta)+p(f\gamma,\gamma)).\end{aligned}$$

It follows that

$$\phi(p(f\alpha,\alpha)+p(f\beta,\beta)+p(f\gamma,\gamma))=0.$$

Thus

$$f\alpha=\alpha, f\beta=\beta \text{ and } f\gamma=\gamma.$$

Hence

$$\alpha=f\alpha=S(\alpha,\beta,\gamma),\beta=f\beta=S(\beta,\gamma,\alpha) \text{ and } \gamma=f\gamma=S(\gamma,\alpha,\beta). \tag{26}$$

Since $S(X\times X\times X)\subseteq g(X)$*, there exist* $r,s,t\in X$ *such that*

$$\alpha=S(\alpha,\beta,\gamma)=gr,\beta=S(\beta,\gamma,\alpha)=gs \text{ and } \gamma=S(\gamma,\alpha,\beta)=gt.$$

Consider

$$\begin{aligned}\psi(\alpha,T(r,s,t)) &=\psi(S(\alpha,\beta,\gamma),T(r,s,t)))\\ &\leq\tfrac{1}{3}\psi(p(f\alpha,gr)+p(f\beta,gs)+p(f\gamma,gt))\\ &\qquad-\phi(p(f\alpha,gr)+p(f\beta,gs)+p(f\gamma,gt))\\ &=\psi(p(\alpha,\alpha)+p(\beta,\beta)+p(\gamma,\gamma))\\ &\qquad-\phi(p(\alpha,\alpha)+p(\beta,\beta)+p(\gamma,\gamma))\\ &=\psi(0)-\phi(0)=0, \quad \text{from } (23),(24) \text{ and } (25).\end{aligned}$$

It follows that $T(r,s,t)=\alpha=gr$.
Similarly $T(s,t,r)=\beta=gs$ *and* $T(t,r,s)=\gamma=gt$.
Since (g,T) *is w - compatible, we have*

$$T(\alpha,\beta,\gamma)=g\alpha,\ T(\beta,\gamma,\alpha)=g\beta \text{ and } T(\gamma,\alpha,\beta)=g\gamma.$$

Now

$$\begin{aligned}\psi(p(\alpha,g\alpha)) &=\psi(p(S(\alpha,\beta,\gamma),T(\alpha,\beta,\gamma)))\\ &\leq\tfrac{1}{3}\psi(p(f\alpha,g\alpha)+p(f\beta,g\beta)+p(f\gamma,g\gamma))\\ &\qquad-\phi(p(f\alpha,g\alpha)+p(f\beta,g\beta)+p(f\gamma,g\gamma))\\ &\leq\tfrac{1}{3}\psi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma))-\phi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma)).\end{aligned}$$

Similarly

$$\psi(p(\beta,g\beta)) \leq\tfrac{1}{3}\psi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma))-\phi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma))$$

and

$$\psi(p(\gamma,g\gamma)) \leq\tfrac{1}{3}\psi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma))-\phi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma)).$$

By definition of ψ, we have

$$\begin{aligned}&\psi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma))\\ &\quad\le\psi(p(\alpha,g\alpha))+\psi(p(\beta,g\beta))+\psi(p(\gamma,g\gamma))\\ &\quad\le\psi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma))-3\phi(p(\alpha,g\alpha)+p(\beta,g\beta)+p(\gamma,g\gamma)).\end{aligned}$$

It follows that

$$\alpha=g\alpha,\beta=g\beta \text{ and } \gamma=g\gamma.$$

Thus

$$\alpha=g\alpha=T(\alpha,\beta,\gamma),\beta=g\beta=T(\beta,\gamma,\alpha) \text{ and } \gamma=g\gamma=T(\gamma,\alpha,\beta). \tag{27}$$

Hence from (26) and (27),
(α,β,γ) is common 3-tupled fixed point of S,T,f and g.
To prove uniqueness,
let $(\alpha^*,\beta^*,\gamma^*)$ is another common 3-tupled fixed point of S,T,f and g.
Consider

$$\begin{aligned}\psi(p(\alpha,\alpha^*))&=\psi(p(S(\alpha,\beta,\gamma),T(\alpha^*,\beta^*,\gamma^*)))\\ &\le\tfrac{1}{3}\psi(p(f\alpha,g\alpha^*)+p(f\beta,g\beta^*)+p(f\gamma,g\gamma^*))\\ &\qquad-\phi(p(f\alpha,g\alpha^*)+p(f\beta,g\beta^*)+p(f\gamma,g\gamma^*))\\ &\le\tfrac{1}{3}\psi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*))-\phi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*)).\end{aligned}$$

Similarly

$$\psi(p(\beta,\beta^*))\le\tfrac{1}{3}\psi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*))-\phi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*))$$

and

$$\psi(p(\gamma,\gamma^*))\le\tfrac{1}{3}\psi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*))-\phi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*)).$$

By definition of ψ, we have

$$\begin{aligned}&\psi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*))\\ &\quad\le\psi(p(\alpha,\alpha^*))+\psi(p(\beta,\beta^*))+\psi(p(\gamma,\gamma^*))\\ &\quad\le\psi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*))-3\phi(p(\alpha,\alpha^*)+p(\beta,\beta^*)+p(\gamma,\gamma^*)).\end{aligned}$$

It follows that

$$\alpha=\alpha^*,\beta=\beta^* \text{ and } \gamma=\gamma^*.$$

Therfore (α,β,γ) is unique common 3-tupled fixed point of S,T,f and g.
Now we claim that $\alpha=\beta=\gamma$.

$$\begin{aligned}\psi(p(\alpha,\beta))&=\psi(p(S(\alpha,\beta,\gamma),T(\beta,\gamma,\alpha)))\\ &\le\tfrac{1}{3}\psi(p(f\alpha,g\beta)+p(f\beta,g\gamma)+p(f\gamma,g\alpha))\\ &\qquad-\phi(p(f\alpha,g\beta)+p(f\beta,g\gamma)+p(f\gamma,g\alpha))\\ &=\tfrac{1}{3}\psi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha))\\ &\qquad-\phi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha)).\end{aligned}$$

$$\begin{aligned}\psi(p(\beta,\gamma)) &= \psi(p(S(\beta,\gamma,\alpha),T(\gamma,\alpha,\beta)))\\ &\le \tfrac{1}{3}\psi(p(f\beta,g\gamma)+p(f\gamma,g\alpha)+p(f\alpha,g\beta))\\ &\qquad -\phi(p(f\beta,g\gamma)+p(f\gamma,g\alpha)+p(f\alpha,g\beta))\\ &= \tfrac{1}{3}\psi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha))\\ &\qquad -\phi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha))\end{aligned}$$

and

$$\begin{aligned}\psi(p(\gamma,\alpha)) &= \psi(p(S(\gamma,\alpha,\beta),T(\alpha,\beta,\gamma)))\\ &\le \tfrac{1}{3}\psi(p(f\gamma,g\alpha)+p(f\alpha,g\beta)+p(f\beta,g\gamma))\\ &\qquad -\phi(p(f\gamma,g\alpha)+p(f\alpha,g\beta)+p(f\beta,g\gamma))\\ &= \tfrac{1}{3}\psi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha))\\ &\qquad -\phi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha)).\end{aligned}$$

By definition of ψ, we have

$$\begin{aligned}&\psi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha))\\ &\quad\le \psi(p(\alpha,\beta))+\psi(p(\beta,\gamma))+\psi(p(\gamma,\alpha))\\ &\quad= \psi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha))-3\phi(p(\alpha,\beta)+p(\beta,\gamma)+p(\gamma,\alpha)).\end{aligned}$$

It follows that $\alpha=\beta=\gamma$.
Thus S,T,f and g have unique common 3-tupled fixed point of the form (α,α,α) in $X\times X\times X$.

Example 3.3 *Let $X=[0,1]$, the mappings $S,T:X\times X\times X\to X$ and $f,g:X\to X$ be defined by $S(x,y,z)=\frac{x^2+y^2+z^2}{6}$, $T(x,y,z)=\frac{x+y+z}{12}$, $f(x)=x^2$ and $g(x)=\frac{x}{2}$ respectively and $p:X\times X\to[0,\infty)$ by $p(x,y)=\max\{x,y\}$. Let $\psi,\phi:[0,\infty)\to[0,\infty)$ be defined by $\psi(x)=x$ and $\phi(x)=\frac{x}{6}$.*
Clearly the conditions (ii),(iii) and (iv) are satisfied.
Now

$$\begin{aligned}p(S(x,y,z),T(u,v,w)) &= \max\left\{\tfrac{x^2+y^2+z^2}{6},\tfrac{u+v+w}{12}\right\}\\ &= \max\left\{\tfrac{x^2}{6}+\tfrac{y^2}{6}+\tfrac{z^2}{6},\tfrac{u}{12}+\tfrac{v}{12}+\tfrac{w}{12}\right\}\\ &\le \max\left\{\tfrac{x^2}{6},\tfrac{u}{12}\right\}+\max\left\{\tfrac{y^2}{6},\tfrac{v}{12}\right\}+\max\left\{\tfrac{z^2}{6},\tfrac{w}{12}\right\}\\ &= \tfrac{1}{6}\left[\max\left\{x^2,\tfrac{u}{2}\right\}+\max\left\{y^2,\tfrac{v}{2}\right\}+\max\left\{z^2,\tfrac{w}{2}\right\}\right]\\ &= \tfrac{1}{6}\left[p(fx,gu)+p(fy,gv)+p(fz,gw)\right]\\ &= \tfrac{1}{3}\left[p(fx,gu)+p(fy,gv)+p(fz,gw)\right]\\ &\qquad -\tfrac{1}{6}\left[p(fx,gu)+p(fy,gv)+p(fz,gw)\right]\\ &= \tfrac{1}{3}\psi(p(fx,gu)+p(fy,gv)+p(fz,gw))\\ &\qquad -\phi(p(fx,gu)+p(fy,gv)+p(fz,gw)).\end{aligned}$$

Hence all conditions of Theorem 3.1 are satisfied and $(0,0,0)$ is unique common 3-tupled fixed point of S,T,f and g.

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