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On Generalized Janowski Starlike Logharmonic

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Abstract

In this paper, we consider univalent logharmonic mappings of the form $f(z) = zh(z)\overline{g(z)}$ defined on the open unit disc with the normalization condition $g(0) = 1$ and $h(0) \neq 0$. We investigate the class of generalized Janowski starlike defined by subordination. Further we obtain the bound for the functional $h(z)/g(z)$ for f in this class.

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1 Introduction

The logharmonic mappings triggered mainly by Z. Abdulhadi and recently few other authors have gained noticeable interest in the current studies of harmonic functions. The basic theory of logharmonic function was developed in [2, 3] in the late 20th century.

Let $H(U)$ be the set of all analytic functions defined on the open unit disc $U = \{z : |z| < 1\}$. A function f is said to be logharmonic on U if it satisfies the nonlinear elliptic partial differential equation

$$\frac{\overline{f_{\overline{z}}}}{\overline{f}} = a\frac{f_z}{f}$$

where the second dilatation function $a \in H(U)$ such that $|a(z)| < 1$ for all $z \in U$. For analytic functions h and g in U, the function f can be expressed as

$$f(z) = h(z)\overline{g(z)}$$

if f is a nonvanishing logharmonic mapping and

$$f(z) = z|z|^{2\beta}h(z)\overline{g(z)}$$

if f vanishes at $z = 0$ but is not identically zero (for $\Re\beta > -\frac{1}{2}$, $g(0) = 1$ and $h(0) \neq 0$).

Let $f(z) = zh(z)\overline{g(z)}$ be a univalent logharmonic mapping where $0 \notin hg(U)$. Then f is starlike logharmonic if

$$\Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > 0.$$

Results on starlike logharmonic of order α was given in [1].

For real numbers A and B, with $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, the family of analytic functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

is in $P(A, B, \alpha)$ if and only if

$$p(z) = \frac{1 + [(1-\alpha)A + \alpha B]\phi(z)}{1 + B\phi(z)}$$

where the function ϕ is analytic in U with $\phi(0) = 0$ and $|\phi(z)| < 1$. The following theorem is also essential for $p(z)$ to be in $P(A, B, \alpha)$.

Lemma 1.1 [5] The function $p(z) \in P(A, B, \alpha)$ if and only if

$$\left| p(z) - \frac{1 - [(1-\alpha)A + \alpha B]Br^2}{1 - B^2r^2} \right| \leq \frac{(1-\alpha)(A-B)r}{1 - B^2r^2}$$

for $|z| = r < 1$.

Let $S^*(A, B, \alpha)$ denote the class of generalized Janowski starlike of the analytic functions $s(z) = z + s_2z^2 + \cdots$ such that $s(z) \in S^*(A, B, \alpha)$ if and only if

$$\frac{zs'(z)}{s(z)} = p(z)$$

and $p(z) \in P(A, B, \alpha)$ for $z \in U$. In the same direction, for univalent logharmonic mapping $f(z) = zh(z)\overline{g(z)}$ with $g(0) = 1$ and $h(0) \neq 0$, we define that f is in the class of generalized Janowski starlike logharmonic denoted by $S^*_{lh}(A, B, \alpha)$ if

$$\left| p(z) - \frac{1 - [(1-\alpha)A + \alpha B]Br^2}{1 - B^2r^2} \right| \leq \frac{(1-\alpha)(A-B)r}{1 - B^2r^2} \tag{1}$$

where

$$p(z) = \frac{h(z)g(z) + zh'(z)g(z) - zg'(z)h(z)}{h(z)g(z)} = 1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}$$

Also, observe that if f satisfy inequality (1), then we have

$$\Re\left(1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right) = \Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) \geq \frac{1 - [(1-\alpha)A + \alpha B]r}{1 - Br}.$$

We consider the above-mentioned class and obtained some interesting results. The following theory of subordination will also be used in our work. For analytic functions F and G, we say F is subordinate to G, written as $F(z) \prec G(z)$, if there is an analytic function ϕ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$, such that $F(z) = G(\phi(z))$, $z \in U$.

2 Main Results

The first result gives the occurrence for which the function in $S^*_{lh}(A, B, \alpha)$ will be starlike.

Theorem 2.1 *The radius of starlikeness for $f(z) = zh(z)\overline{g(z)} \in S^*_{lh}(A, B, \alpha)$ is*

$$r = \frac{1}{(1-\alpha)A + \alpha B}.$$

Proof 1 *For $f \in S^*_{lh}(A,B,\alpha)$, we have*

$$\Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) \geq \frac{1-[(1-\alpha)A+\alpha B]r}{1-Br}. \tag{2}$$

Also, for f to be starlike, we need

$$\Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) > 0. \tag{3}$$

From (2) and (3), we see that

$$\Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) \geq \frac{1-[(1-\alpha)A+\alpha B]r}{1-Br} > 0$$

is true if

$$r < \frac{1}{(1-\alpha)A+\alpha B}.$$

Corollary 2.2 *If $f(z) = zh(z)\overline{g(z)} \in S^*_{lh}(A,B)$, then f is starlike log-harmonic of order α in $|z| \leq r$ where*

$$r = \frac{1-\alpha}{A-\alpha B}.$$

Proof 2 *The proof is complete by solving*

$$\Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) \geq \frac{1-Ar}{1-Br} > \alpha.$$

Theorem 2.3 *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping on U. If*

$$\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec \frac{(1-\alpha)(A-B)z}{1+Bz} \tag{4}$$

then $f \in S^*_{lh}(A,B,\alpha)$.

Proof 3 *For $f \in S^*_{lh}(A,B,\alpha)$, we see that*

$$\Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) \geq \frac{1-[(1-\alpha)A+\alpha B]}{1-B}.$$

and the above is true if

$$1 + \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} = p(z) = \frac{1+[(1-\alpha)A+\alpha B]\phi(z)}{1+B\phi(z)}$$

or if

$$\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} = \frac{(1-\alpha)A\phi(z)+\alpha B\phi(z)-B\phi(z)}{1+B\phi(z)}.$$

For $\alpha = 0$, we have the following result.

Corollary 2.4 [4] *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping on U. If*

$$\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec \frac{(A-B)z}{1+Bz}$$

then $f \in S^*_{lh}(A, B)$.

For $A = 1,\ B = -1$, we have the result for the function $f(z) = zh(z)\overline{g(z)}$ in $S^*_{lh}(\alpha)$, the class of starlike log-harmonic of order α defined by subordination.

Corollary 2.5 *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping on U. If*

$$\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec \frac{2(1-\alpha)z}{1-z}$$

then $f \in S^*_{lh}(\alpha)$.

When $A = 1,\ B = -1,\ \alpha = 0$, the result will reduce to the usual starlike logharmonic class defined by subordination.

Corollary 2.6 [6] *Let $f(z) = zh(z)\overline{g(z)}$ be a logharmonic mapping on U. If*

$$\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)} \prec \frac{2z}{1-z}$$

then $f \in S^*_{lh}$.

Theorem 2.7 *If $f(z) = zh(z)\overline{g(z)} \in S^*_{lh}(A, B, \alpha)$, then*

$$(1-Br)^{\frac{(1-\alpha)(A-B)}{B}} \leq \left|\frac{h(z)}{g(z)}\right| \leq (1+Br)^{\frac{(1-\alpha)(A-B)}{B}} \qquad for \quad B \neq 0$$

and

$$e^{-(1-\alpha)Ar} \leq \left|\frac{h(z)}{g(z)}\right| \leq e^{(1-\alpha)Ar} \qquad for \quad B = 0.$$

Proof 4 *From (1), for $B \neq 0$ we have*

$$\frac{1-[(1-\alpha)A+\alpha B]r}{1-Br} \leq \Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) \leq \frac{1+[(1-\alpha)A+\alpha B]r}{1+Br}$$

and hence

$$\frac{-(1-\alpha)(A-B)r}{1-Br} \leq \Re\left(\frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}\right) \leq \frac{(1-\alpha)(A-B)r}{1+Br}. \qquad (5)$$

Rewriting inequality (5) yields

$$\frac{-(1-\alpha)(A-B)}{1-Br} \leq \frac{\partial}{\partial r}\log\left|\frac{h(z)}{g(z)}\right| \leq \frac{(1-\alpha)(A-B)}{1+Br}$$

which leads to the desire result.

For $B = 0$, (1) yields

$$1-(1-\alpha)Ar \leq \Re\left(\frac{zf_z - \overline{z}f_{\overline{z}}}{f}\right) \leq 1+(1-\alpha)Ar$$

and the proof is completed similarly.

Corollary 2.8 [4] *If $f(z) = zh(z)\overline{g(z)} \in S^*_{lh}(A, B)$, then*

$$(1-Br)^{\frac{(A-B)}{B}} \leq \left|\frac{h(z)}{g(z)}\right| \leq (1+Br)^{\frac{(A-B)}{B}} \qquad for \quad B \neq 0$$

and

$$e^{-Ar} \leq \left|\frac{h(z)}{g(z)}\right| \leq e^{Ar} \qquad for \quad B = 0.$$

Corollary 2.9 *If $f(z) = zh(z)\overline{g(z)} \in S^*_{lh}(\alpha)$, then*

$$(1+r)^{2\alpha-2} \leq \left|\frac{h(z)}{g(z)}\right| \leq (1-r)^{2\alpha-2}.$$

Corollary 2.10 *If $f(z) = zh(z)\overline{g(z)} \in S^*_{lh}$, then*

$$\frac{1}{(1+r)^2} \leq \left|\frac{h(z)}{g(z)}\right| \leq \frac{1}{(1-r)^2}.$$

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