Mathematica Aeterna, Vol. 1, 2011, no. 08, 537 - 545

Design and Analysis of Matrices for Two Dimensional Cellular Automata Linear rules in Hexagonal Neighborhood

Jahangir Mohammed

Institute of Mathematics and Applications
Andharua, Bhubaneswar-751003, India
jahangir.isi@gmail.com

Sudhakar Sahoo

Institute of Mathematics and Applications
Andharua, Bhubaneswar-751003, India
sudhakar.sahoo@gmail.com

Abstract

Constructing a binary matrix for a linear Cellular Automata (CA) rule is useful for finding the successor state of a CA. There are applications where the nearest neighborhood structure in a hexagonal grid is found to be simpler than that of a rectangular grid. In this paper, binary matrices are considered for modeling two-dimensional hexagonal grids with nearest neighborhood structure. This gives 128 number of two-state, null boundary, uniform, linear hexagonal CA rules. These hexagonal CA rules are classified into different groups, among them the list for reversible rules, self symmetric rules and pair wise symmetric rules are found using matrix properties.

Mathematics Subject Classification: 37B15, 37-XX

Keywords: Cellular Automata, Hexagonal Neighborhood, Linear Rules, Null Boundary Condition, Rule Matrix.

1 Introduction

A Cellular Automaton (CA) is a mathematical structure, modeling a set of cells which interact with their neighbors. In this structure, each cell have values known as states, all the cells update their states simultaneously at discrete time steps, and the new state of a cell is determined by current state of its neighbors according to a local function called rule of the CA [6]. Hexagonal grid CA is a tessellation of the plane by regular hexagons which provide for higher packing density of cells and give a more accurate approximation of circular regions than that of CA with rectangular grids. Furthermore the unit cells of hexagonal grids are uniformly connected in the sense that the distance from a given cell to any adjacent cell is the same [1]. Previously, using matrix algebra 2-D, linear CA rules were studied on the domain of nine-neighborhood rectangular co-ordinate system [3-5]. Here, in this paper, we have studied 2-D, linear CA rules on seven-neighborhood hexagonal co-ordinate system.

2 Mathematical Model for 2-D CA in Hexagonal Neighborhood

2.1 Co-ordinate system for a hexagonal grid

Hexagonal grid is slightly different from the usual rectangular grid. The irregular rectangular co-ordinate system, with the two co-ordinates row number and position in row (Fig. 1), may be the simplest way to label hexagons [2]. So, we can define the dimension of a hexagonal grid as $(m \times n)$ when m is the total number of rows in the hexagonal grid and n is the number of columns in a row that present in the hexagonal grid.

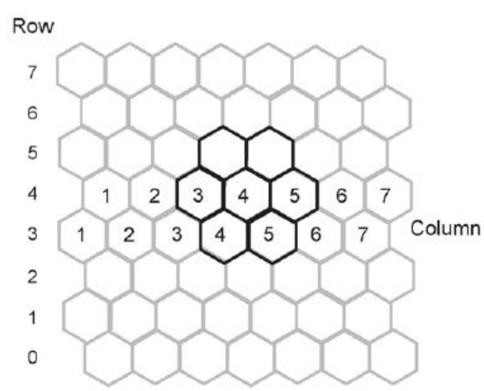

Fig. 1. A row and column co-ordinate system for a hexagonal grid

2.2 Naming scheme and hexagonal linear CA rules

The naming scheme adopted in [3-5] for rectangular nine-neighborhood CA is modified in case of hexagonal seven-neighborhood CA as shown in fig 2. In 2-D seven neighborhoods hexagonal CA the next state of a particular cell is determined by the current state of itself and six other cells in its nearest neighborhood. Such dependencies are accounted by various rules. The central

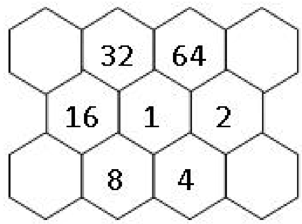

Fig. 2. Hexagonal structure and its basic rules

box represents the current cell (i.e. the cell being considered) and all other boxes represent the six nearest neighbors of that cell. The number within each box represents the rule number characterizing the dependency of the current cell on that particular neighbor only. $Rule_1$ characterizes dependency of the central cell on itself alone. $Rule_2$ characterizes dependency of the right neighbor to the central cell and so on.
These seven rules 1, 2, 4, 8, 16, 32 and 64 are called fundamental rules. In case the cell has dependency on two or more neighboring cells, the rule number will be the arithmetic sum of the numbers of the relevant cells. For example, the 2-D CA rule 127 (64+32+16+8+4+2+1) refers to the seven-neighborhood dependency of the central cell on (top left, top right, bottom left, bottom right, left , right and itself). The number of such rules is ${}^7C_0 + {}^7C_1 + {}^7C_2 + ... + {}^7C_7 = 2^7 = 128$ which includes rule characterizing no dependency.

2.3 Characterizing 2-D Liner Hexagonal CA using matrices and logical XOR ($\bigoplus$) operator

Let L be the set of 128 hexagonal linear CA rules (Same as boolean functions [5]). We can define the logical operation XOR denoted by $\oplus$ on the set of linear CA rules as $Rule_m \oplus Rule_n = Rule_k$, Where if the binary representations of m, n and k are $m_1m_2....m_7, n_1n_2....n_7$ and $k_1k_2....k_7$, then $k_i = m_i \oplus n_i$, for i =1, 2,...7. Clearly $Rule_m \oplus Rule_m = Rule_0$ and $Rule_m \oplus Rule_0 = Rule_m = Rule_0 \oplus Rule_m$.
Thus under the operation $\oplus$, L is a commutative group where each member is its own inverse. Any member L can be uniquely expressed as the sum of the fundamental linear rules as shown in the following example.

Example 2.1: For finding the expression for $Rule_{25}$ in terms of the fundamental rules the following steps has to be followed.
Step1:- Find the binary representation of 25 i.e., $(25)_{10} = (0011001)_2$.
Step2:- Here 0^{th} bit, 3^{rd} bit and 4^{th} bit from the least significant bit position are non-zero.
Step3:- The place values for these non-zero bit positions are
$1 \times 2^0 = 1$, $1 \times 2^3 = 8$ and $1 \times 2^4 = 16$ respectively.

Step4:- Thus, $Rule_{25}$ is the combination of three fundamental rules $Rule_1$, $Rule_8$ and $Rule_{16}$ i.e., $Rule_{25} = Rule_1 \oplus Rule_8 \oplus Rule_{16}$

3 A simple procedure to design rule matrices for 2-D, nearest neighborhood, 128, Hexagonal, Uniform, linear CA rules in null boundary condition

In null boundary condition the neighbors of the boundary cells those are present outside the hexagonal grid region are all treated as state zero.

Theorem 3.1 *If a hexagonal grid is of dimension* $(m \times n)$ *then the rule matrix has dimension* $(mn \times mn)$.

Proof: From the construction procedure using basis matrices given in [3-5] it has been found that every element of the problem matrix will correspond to a column vector of the rule matrix. As 'mn' number of elements present in a $(m \times n)$ problem matrix, so the number of column vectors in the rule matrix will be 'mn'. Again each column vector is of dimension $(mn \times 1)$, so the dimension of the rule matrix is $(mn \times mn)$. Hence proved.

Let M_i denote the Rule matrix for $Rule_i$, for i =0, 1, 2, 3,....., 127. So, M_1 = rule matrix for $Rule_1$, M_2 = rule matrix for $Rule_2$ and so on.

Next, on using basis matrices (similar to the procedure in [3-5]) we have constructed a rule matrix for $Rule_2$ of a hexagonal problem matrix of dimension (3×3) as shown in Example 3.1.

Example-3.1: Let a hexagonal grid, two dimensional problem matrix is of order (3×3) and let us apply $Rule_2$ into it.

$$\begin{pmatrix} 1 & & 0 & & 1 & \\ & 1 & & 0 & & 1 \\ 1 & & 1 & & 0 & \end{pmatrix}_{(3\times3)} \longrightarrow \begin{pmatrix} 0 & & 1 & & 0 & \\ & 0 & & 1 & & 0 \\ 1 & & 0 & & 0 & \end{pmatrix}_{(3\times3)}$$

From Theorem 3.1, the hexagonal rule matrix for $Rule_2$ i.e., M_2 has dimension (9×9) as shown below.

$$M_2 = \begin{pmatrix} 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{(9\times 9)}$$

This rule matrix when multiplied with the problem matrix arranged in a column vector will produce the next state of the automaton.

It has been observed that for three basic rules (2, 4 and 8), five specific type of binary sequences namely s_1, s_2, s_3, s_4 and s_5 are found in different diagonals of their rule matrices.These sequences are as follows
$s_1 = \underbrace{111\ldots1}_{n-1}0\underbrace{111\ldots1}_{n-1}0\ldots\ldots = 1^{n-1}01^{n-1}0\ldots\ldots$
This sequence s_1 contains $(n-1)$ number of 1's and a 0 then $(n-1)$ number of 1's follows a 0 and then this specific pattern is repeated and finally ends with $(n-1)$ number of 1's i.e., the sequence s_1 contains m number of subsequences of the form 1^{n-1} which are separated by m number of 0s. And
$s_2 = 1^n0^n1^n0^n\ldots\ldots$
$s_3 = 0^n1^{n-1}0^{n+1}1^{n-1}0^{n+1}\ldots\ldots$
$s_4 = 01^{n-1}0^{n+1}1^{n-1}0^{n+1}\ldots\ldots$
$s_5 = 0^n1^n0^n1^n\ldots\ldots$

Theorem 3.2 *The four basic rule matrices possess a particular structure as follows.*
M_1 *is the identity matrix i.e., the diagonal sequence is of the form* 1^n *and all other elements are zero.*
M_2 = *Super diagonal position is the sequence* s_1 *and all other elements are zero.*
$M_4 = (n+1)^{th}$ *and* $(n+2)^{th}$ *diagonal positions are the sequences* s_2 *and* s_3 *respectively and all other elements are zero.*
$M_8 = n^{th}$ *and* $(n+1)^{th}$ *diagonal positions are the sequence* s_4 *and* s_5 *respectively and all other elements are zero.*

We have examined all linear CA rules by taking a hexagonal grid problem matrix of dimension (10×10) and therefore by Theorem 3.1 its rule matrix has dimension (100×100). As described in Theorem-3.2, the binary images for the rule matrices M_2, M_4 and M_8 are shown in the figures 3 to 5. In all this binary image, '0' pixel value represents black and '1' pixel value represents white. In fig. 3 white dotted lines represent the sequences of 1's in the super diagonal position and all other pixels are black means that, they are all 0's.

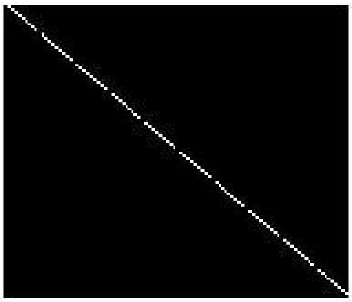

Fig. 3. M_2 matrix

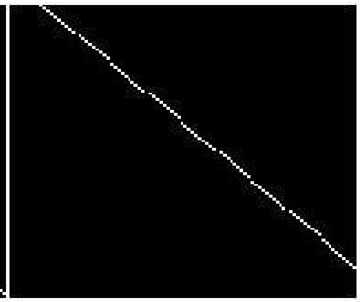

Fig. 4. M_4 matrix

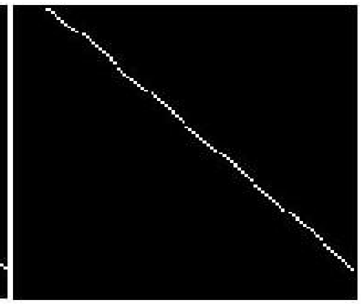

Fig. 5. M_8 matrix

Theorem 3.3 *Matrices corresponding to the fundamental CA rules in hexagonal neighborhood are related to each other in the following manner.*
M_1 *= Identity matrix.*
$(M_2)^T = M_{16}$ *i.e. transpose of* $Rule_2$ *matrix is* $Rule_{16}$ *matrix.*
$(M_4)^T = M_{32}$ *and* $(M_8)^T = M_{64}$.
And other rule matrices can be generated from these 4 basic rule matrices by the help of two operations, one is the transpose denoted by T, another is $\oplus$.

Example-3.1: If $M = \{M_1, M_2, M_4, M_8\}$ be the set of 4 basic matrices then
$M_9 = M_1 \oplus M_8$.
$M_{69} = M_{64} \oplus M_4 \oplus M_1 = (M_8)^T \oplus M_4 \oplus M_1$.
$M_{90} = M_{64} \oplus M_{16} \oplus M_8 \oplus M_2 = (M_8)^T \oplus (M_2)^T \oplus M_8 \oplus M_2$.

The binary images for hexagonal rule matrices as mentioned in Example-3.1 are shown from figures 6 to 8. In matrix M_9 three different white dotted lines for three sequences s_1, s_4 and s_5 are visible in the 1^{st}, 11^{th} and 12^{th} diagonal positions, that shows $Rule_9$ is the addition of two fundamental hexagonal CA rules. Similarly the matrices M_{69} and M_{90} are consisting of three and four fundamental rules respectively.

All the above images are prepared using MATLAB version 7.6.0(2008a).

4 Classification of CA rules using matrix algebra

4.1 Classification

By taking XOR operations of different fundamental rules we have classified the rules into different groups. For example, In group 2, XOR operation is

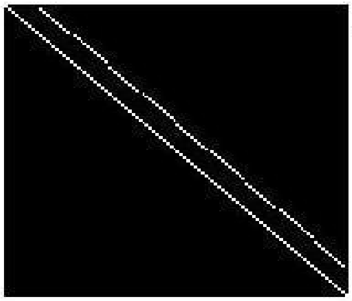

Fig. 6. M_9 matrix

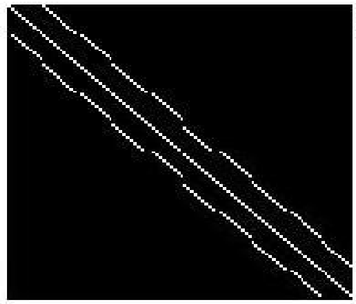

Fig. 7. M_{69} matrix

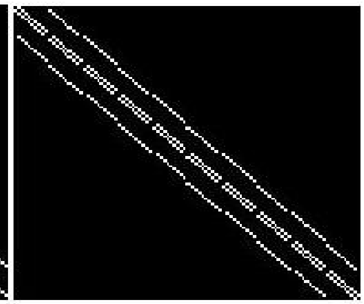

Fig. 8. M_{90} matrix

taking place using two fundamental rules and in group 3, XOR operations are taking place using three fundamental rules.

All these 127 except $Rule_0$, two-dimensional, hexagonal, uniform, linear CA rules has been arranged in the following seven groups

Group 1 rules : 1, 2, 4, 8, 16, 32 and 64.

Group 2 rules : 3, 5, 6, 9, 10, 12, 17, 18, 20, 24, 33, 34, 36, 40, 48, 65, 66, 68, 72, 80 and 96.

Group 3 rules : 7, 11, 13, 14, 19, 21, 22, 25, 26, 28, 35, 37, 38, 41, 42, 44, 49, 50, 52, 56, 67, 69, 70, 73, 74, 76, 81, 82, 84, 88, 97, 98, 100, 104 and 112.

Group 4 rules : 15, 23, 27, 29, 30, 39, 43, 45, 46, 51, 53, 54, 57, 58, 60, 71, 75, 77, 78, 83, 85, 86, 89, 90, 92, 99, 101, 102, 105, 106, 108, 113, 114, 116 and 120.

Group 5 rules : 31, 47, 55, 59, 61, 62, 79, 87, 91, 93, 94, 103, 107, 109, 110, 115, 117, 118, 121, 122 and 124.

Group 6 rules : 63, 95, 111, 119, 123, 125 and 126.

Group 7 rule : 127.

4.2 Non-singular Rule matrix (Reversible CA)

Since Rule matrices are necessarily square matrices, the non-singular rule matrix called reversible CA deserves to be studied closely because a non-singular matrix must produce cyclic state transition diagram having enumerable applications in Pattern Classification, Clustering, Encryption and Decryption, Data Compression etc.

Out of 128 linear rule matrices, following 15 rules are always reversible for any hexagonal grid problem matrix of dimension $(m \times n)$.

Group 1 rules : 1.

Group 2 rules : 3, 5, 9, 17, 33, 65.

Group 3 rules : 7, 11, 13, 49, 81, 97.

Group 4 rules : 15, 113.

4.3 Self-Symmetric and Pairwise symmetric Rules

Self-symmetric CA rules are those linear rules whose matrices are their own transpose whereas a CA rule whose transpose matrix is the matrix for another linear CA rule is called as pairwise symmetric rule. Out of 128 linear CA rules, the number of self-symmetric CA rules are 16 which include the $Rule_0$. The list of all these rules except $Rule_0$ in different groups are given below
Group 1 rules : 1.
Group 2 rules : 18, 36, 54, 72.
Group 3 rules : 19, 37, 55, 73.
Group 4 rules : 90, 108.
Group 5 rules : 91, 109.
Group 6 rule : 126.
Group 7 rule : 127.

It has been observed that the rest 112 linear CA rules are divided into 56 number of pairwise symmetric rules as follows
(2, 16), (3, 17), (4, 32), (5, 33), (6, 48), (7, 49), (8, 64), (9, 65), (10, 80), (11, 81), (12, 96), (13, 97), (14, 112), (15, 113), (20, 34), (21, 35), (22, 50), (23, 51), (24, 66), (25, 67), (26, 82), (27, 83), (28, 98), (29, 99), (30, 114), (31, 115), (38, 52),(39, 53), (40, 68), (41, 69), (42, 84), (43, 85), (44, 100), (45, 101), (46, 116), (47, 117), (56, 70), (57, 71), (58, 86), (59, 87), (60, 102), (61, 103), (62, 118), (63, 119), (74, 88), (75, 89), (76, 104), (77, 105), (78, 120), (79, 121), (92, 106), (93, 107), (94, 122), (95, 123), (110, 124), (111, 125).

5 Conclusion

The seven-neighborhood hexagonal grid, CA is a better and closer alternative than that of nine-neighborhood rectangular grids. Here only 128 uniform linear rules are possible which is 75% less in number than that of nine-neighborhoods uniform linear rules. Finding the structure of hexagonal rule matrices help us to study the behavior of different hexagonal CA rules. If we can study enough properties of hexagonal rule matrices then many unanswered question on CA in the hexagonal domain can be settled. Although the findings are only for null boundary conditions, similar study can also be done for other boundary conditions such as fixed, periodic, reflexive, adiabatic etc. The concept of graph theory by treating the rule matrices as adjacency matrix will be our immediate future research goal. Another area of concern is to reduce the size of the rule matrix to the size of its problem matrix so that it can be easily studied.

ACKNOWLEDGEMENTS. Authors would like to acknowledge Prof. S. Pattanayak, Institute of Mathematics and Applications, Prof. B. K. Nayak,

P. G. Dept. of Mathematics, Utkal University and Prof. P. Pal Choudhury, Indian Statistical Institute, Kolkata for their constant encouragement and support.

References

[1] Allen, J. D., *Filter banks for images on hexagonal grid*, Signal Solutions, 2003.

[2] Birch, C. P. D., Oom, S. P., Beecham, J. A., *Rectangular and hexagonal grids used for observation, experiment and simulation in ecology*, Ecol. Model 206, 2007, 347-359.

[3] Khan, A. R., Choudhury, P. P., Dihidar, K., Mitra, S., Sarkar, P., *VLSI architecture of cellular automata machine*, Computers and Mathematics with Applications, 1997, 33(5), 79-94.

[4] Dihidar, K., Choudhury, P. P., *Matrix Algebraic formulae concerning some exceptional rules of two-dimensional cellular automata*, Information Sciences, 2004, 165(1-2), 91-101.

[5] Choudhury, P. P., Nayak, B. K., Sahoo, S., Rath, S. P., *Theory and Applications of Two-dimensional, Null-boundary, Nine-Neighborhood,Cellular Automata Linear rules*, arXiv:0804.2346, cs.DM;cs.CC; cs.CV, 2008.

[6] Wolfram, S., *A New Kind of Science*, Wolfram Media, Inc, 2002.

Received: October 28, 2011