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Global Analysis of an SVEIR Epidemic Model with Partial Immunity

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Abstract

In this paper, an SVEIR epidemic model with nonlinear incidence rate are established under the assumption that the vaccinated individuals have partial immunity, and the basic productive number is obtained according to the next generation matrix. By Liapunov-Lasalle invariant theorem, the globally asymptotical stability of the disease-free equilibrium is proved. By Hurwitz criterion, the local asymptotic stability of the endemic equilibrium was proved, The sufficient conditions for the globally asymptotically stable of the endemic equilibrium are obtained.

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1 Introduction

Infectious diseases have tremendous influence on human life, mathematical models describing the population dynamics of infectious diseases have been playing an important role in better understanding of epidemiological patterns and disease control for a long time. In order to predict the spread of infectious disease among regions, many epidemic models have been proposed and analyzed in recent years[$1-4$]. But many diseases such as measles, severe acute respiratory syndromes(SARS) and so on, however, incubate inside the hosts for a period of time before the hosts become infectious. So it is necessary to investigate the role of incubation in disease transmission. Mathematical models with latent period are numerous in the literature (see [5-8]). The latent time delay is incorporated into the SEIR model by Yan and Liu [9].

Vaccination is one of commonly used method for predicting and controlling disease spread. The epidemic models with vaccination have been investigated recently by some authors[$10-15$]. They[14,15]assume that a susceptible individual goes through a latent period after infection before becoming infectious, they established SEIV and SEIR epidemic models with nonlinear incidence rates and discussed stability of equilibrium point, respectively. But these articles all assumed that the vaccinees obtained the immunity fully, As far as we know, it is hard to obtain the immunity fully for the vaccinees, so, in Ref.[16], partial immunity was considered.

In this paper, incorporating a general nonlinear incidence rate and a waning preventive vaccines, we consider a model with a nonlinear incidence rate, it is assumed that the vaccinees obtain only partial immunity, and a latent period is also taken into account.That is, we consider the following system:

$$\begin{cases} \frac{dS}{dt} = (1-p)A - \mu S - \frac{\beta SI}{\varphi(S)} + \gamma V, \\ \frac{dV}{dt} = pA - \sigma\beta VI - (\mu+\gamma)V, \\ \frac{dE}{dt} = \frac{\beta SI}{\varphi(S)} + \sigma\beta VI - (\mu+\epsilon)E, \\ \frac{dI}{dt} = \epsilon E - (\mu+\delta+\alpha)I, \\ \frac{dR}{dt} = \delta I - \mu R. \end{cases} \tag{1}$$

where $S = S(t), V = V(t), E = E(t), I = I(t)$ and $R = R(t)$ denote the susceptible, vaccinated, exposed, infectious and recovered individuals at time t, respectively. A is the constant recruitment rate of individuals, and death rate for disease and natural death rate are α and μ , respectively. Let β be the transmission rate of disease when susceptible individuals are contact with infected individuals. p is the fraction of recruited individuals who are vaccinated, γ is the rate at which vaccine wanes, ϵ is the rate at which exposed individuals become infectious, the recovery rate of infected individuals is δ , the vaccinees who contact infected individuals before obtaining immunity have the possibility of infection with a disease transmission rate $\sigma\beta$ $(0 \leq \sigma \leq 1)$, $\sigma = 0$ denotes that the vaccinees obtained the full immunity, $\sigma = 1$ denotes that vaccine failed in work fully. It is assumed that the vaccinees obtain partial immunity, that is to say, $0 < \sigma < 1$. The nonlinear incidence is assumed to be of the form $\frac{\beta SI}{\varphi(S)}$, we assume that function $\varphi(S)$ satisfies: $\varphi(0) = 1$, $(\frac{S}{\varphi(S)})' > 0$.

The paper is organized as follows. In section 2, the existence of equilibria is discussed. In Section 3, the stability of equilibria is investigated. In Section 4, the persistence of system (2) is discussed. In Section 5, global asymptotic stability of the endemic equilibrium is also investigated. The paper ends up with brief remarks.

2 Existence of equilibria

In this section, we will discuss the existence of the disease-free equilibrium and the endemic equilibrium of the model (1). Since the equation for R is independent from other equations, we have the following sub system

$$\begin{cases} \frac{dS}{dt} = (1-p)A - \mu S - \frac{\beta SI}{\varphi(S)} + \gamma V, \\ \frac{dV}{dt} = pA - \sigma\beta VI - (\mu+\gamma)V, \\ \frac{dE}{dt} = \frac{\beta SI}{\varphi(S)} + \sigma\beta VI - (\mu+\epsilon)E, \\ \frac{dI}{dt} = \epsilon E - (\mu+\delta+\alpha)I. \end{cases} \tag{2}$$

From the reduced model (2), we have

$$\frac{d(S+V+E+I)}{dt} = A - \mu(S+V+E+I) - (\delta+\alpha)I \leq A - \mu(S+V+E+I),$$

then

$$\lim_{t\to+\infty} \sup(S+V+E+I) \leq \frac{A}{\mu}.$$

It is easy to know that,

$$\Omega = \{ (S,V,E,I) \in R_+^4 \mid S+V+E+I \leq \frac{A}{\mu}\},$$

is a positively invariant region for model (2) , and model (2) is obviously well-posed in Ω as follows.

It is easy to check that model (2) always has the disease-free equilibrium $P_0(S_0, V_0, 0, 0)$, where $S_0 = \frac{A}{\mu} - \frac{Ap}{\mu+\gamma}$, $V_0 = \frac{Ap}{\mu+\gamma}$.

To consider the existence and uniqueness of endemic equilibrium $P^*(S^*, V^*, E^*, I^*)$,we firstly study the basic reproductive number R_0 of model (2) according to the next generation matrix[2].

Let $X = (E, I, S, V)^T$. So model (2) can be written as $\frac{dX}{dt} = \mathcal{F}(X) - \mathcal{V}(X)$, where

$$\mathcal{F}(X) = \begin{pmatrix} \frac{\beta SI}{\varphi(S)} + \sigma\beta VI \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{V}(X) = \begin{pmatrix} (\mu+\epsilon)E \\ (\mu+\delta+\alpha)I - \epsilon E \\ \mu S + \frac{\beta SI}{\varphi(S)} - \gamma V - (1-p)A \\ \sigma\beta VI + (\mu+\gamma)V - pA \end{pmatrix}.$$

So,

$$D\mathcal{F}(P_0) = \begin{pmatrix} \mathcal{F}_{2\times2} & 0 \\ 0 & 0 \end{pmatrix}, \quad D\mathcal{V}(P_0) = \begin{pmatrix} & \mathcal{V}_{2\times2} & & 0_{2\times2} \\ 0 & \frac{\beta S_0}{\varphi(S_0)} & \mu & -\gamma \\ 0 & \sigma\beta V_0 & 0 & \mu+\gamma \end{pmatrix},$$

where,

$$\mathcal{F}_{2\times 2} = \begin{pmatrix} 0 & \frac{\beta S_0}{\varphi(S_0)} + \sigma\beta V_0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{V}_{2\times 2} = \begin{pmatrix} \mu+\epsilon & 0 \\ -\epsilon & \mu+\delta+\alpha \end{pmatrix}.$$

so spectral radius of the next generation matrix $\mathcal{F}\mathcal{V}^{-1}$ can be found as,

$$\rho(\mathcal{F}\mathcal{V}^{-1}) = \frac{\epsilon\beta(\frac{S_0}{\varphi(S_0)} + \sigma V_0)}{(\mu+\epsilon)(\mu+\delta+\alpha)}.$$

Thus, the basic reproductive number R_0 of the model (2) can be found as

$$R_0 = \frac{\epsilon\beta(\frac{S_0}{\varphi(S_0)} + \sigma V_0)}{(\mu+\epsilon)(\mu+\delta+\alpha)}.$$

Endemic equilibrium $P^*(S^*, V^*, E^*, I^*)$ of the model (2) can be determined by the following equations

$$\begin{cases} (1-p)A - \mu S - \frac{\beta SI}{\varphi(S)} + \gamma V = 0, \\ pA - \sigma\beta VI - (\mu+\gamma)V = 0, \\ \frac{\beta SI}{\varphi(S)} + \sigma\beta VI - (\mu+\epsilon)E = 0, \\ \epsilon E - (\mu+\delta+\alpha)I = 0. \end{cases} \tag{3}$$

From the forth equation of (3), we obtain $E = \frac{\mu+\delta+\alpha}{\epsilon}I$, substituting it into the third equation of (3), we obtain the following equation

$$\frac{\beta S}{\varphi(S)} + \sigma\beta V = \frac{(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon},$$

$$V = \frac{\frac{(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon} - \frac{\beta S}{\varphi(S)}}{\sigma\beta}.$$

From the second equation of model (3), we obtain

$$I = \frac{pA - (\mu+\gamma)V}{\sigma\beta V} = \frac{pA}{\sigma\beta V} - \frac{\mu+\gamma}{\sigma\beta}.$$

After substituting V and I into the first equation of model (3), we obtain the following equation for S

$$(1-p)A - \mu S - \frac{\beta S(\frac{pA}{\sigma\beta V} - \frac{\mu+\gamma}{\sigma\beta})}{\varphi(S)} + \gamma(\frac{(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon\sigma\beta} - \frac{S}{\sigma\varphi(S)}) = 0.$$

After some algebraic calculation, we have

$$A - \mu S + \frac{pA(\mu+\epsilon)(\mu+\delta+\alpha)}{\frac{\epsilon\beta S}{\varphi(S)} - (\mu+\epsilon)(\mu+\delta+\alpha)} + \frac{\mu S}{\sigma\varphi(S)} + \frac{\gamma(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon\sigma\beta} = 0.$$

Let

$$F(S) = A - \mu S + \frac{pA(\mu+\epsilon)(\mu+\delta+\alpha)}{\frac{\epsilon\beta S}{\varphi(S)} - (\mu+\epsilon)(\mu+\delta+\alpha)} + \frac{\mu S}{\sigma\varphi(S)} + \frac{\gamma(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon\sigma\beta}.$$

It can easily seen that $F(0) > 0$. Next, we determine the sign of $F^{'}(S)$:

$$\begin{aligned} F'(S) = & -\mu - \frac{pA\epsilon\beta(\mu+\epsilon)(\mu+\delta+\alpha)(\frac{S}{\varphi(S)})'}{(\frac{\epsilon\beta S}{\varphi(S)}-(\mu+\epsilon)(\mu+\delta+\alpha))^2} + \frac{\mu}{\sigma}(\frac{S}{\varphi(S)})' \\ < & -\mu + \frac{pA\epsilon\beta(\frac{S}{\varphi(S)})'}{\frac{\epsilon\beta S}{\varphi(S)}-(\mu+\epsilon)(\mu+\delta+\alpha))} + \frac{\mu}{\sigma}(\frac{S}{\varphi(S)})' \\ = & -\mu + (\frac{\mu}{\sigma} - \frac{pA\beta}{\frac{(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon} - \frac{\beta S}{\varphi(S)}})(\frac{S}{\varphi(S)})' < 0. \end{aligned}$$

Moreover, if $R_0 > 1$, then $\epsilon\beta(\frac{S_0}{\varphi(S_0)} + \frac{\sigma Ap}{\mu+\gamma}) > (\mu+\epsilon)(\mu+\delta+\alpha)$.

$$\begin{aligned} F(S_0) = & A - \mu S_0 + \frac{pA(\mu+\epsilon)(\mu+\delta+\alpha)}{\frac{\epsilon\beta S_0}{\varphi(S_0)}-(\mu+\epsilon)(\mu+\delta+\alpha)} + \frac{\mu S_0}{\sigma\varphi(S_0)} + \frac{\gamma(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon\beta\sigma} \\ < & A - \mu S_0 - \frac{(\mu+\gamma)(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon\beta\sigma} + \frac{\mu S_0}{\sigma\varphi(S_0)} + \frac{\gamma(\mu+\epsilon)(\mu+\delta+\alpha)}{\epsilon\beta\sigma} \\ < & A - \mu S_0 - \frac{\mu S_0}{\sigma\varphi(S_0)} - \frac{\mu pA}{\mu+\gamma} + \frac{\mu S_0}{\sigma\varphi(S_0)} = A - \mu S_0 - \frac{\mu pA}{\mu+\gamma} = 0. \end{aligned}$$

Therefore the unique root of the equation $F(S) = 0$ always exists in $(0, S_0)$. If $S > S_0$, $F(S) < 0$. So S^* is the unique positive root of the equation $F(S) = 0$.

That is to say, if $R_0 \leq 1$, model (2) only has the disease-free equilibrium $P_0(S_0, V_0, 0, 0)$; if $R_0 > 1$, there is a unique endemic equilibrium $P^*(S^*, V^*, E^*, I^*)$ except for the disease-free equilibrium P_0.

3 Stability of equilibria

In this section, we will discuss the stability of the disease-free equilibrium and the endemic equilibrium of the model (1).

In the following, Firstly we investigate the stability of disease-free equilibrium P_0. The Jacobian matrix of model (2) at the disease-free equilibrium P_0 is as follows

$$\begin{pmatrix} -\mu & \gamma & 0 & -\frac{\beta S_0}{\varphi(S_0)} \\ 0 & -\mu-\gamma & 0 & -\sigma\beta V_0 \\ 0 & 0 & -\mu-\epsilon & \sigma\beta V_0 + \frac{\beta S_0}{\varphi(S_0)} \\ 0 & 0 & \epsilon & -(\mu+\alpha+\delta) \end{pmatrix}.$$

So the corresponding characteristic equation is

$$(\lambda+\mu)(\lambda+\mu+\gamma)[(\lambda+\mu+\epsilon)(\lambda+\mu+\delta+\alpha) - \epsilon(\frac{\beta S_0}{\varphi(S_0)} + \sigma\beta V_0)] = 0. \quad (4)$$

It is easy to see that characteristic equation (4) always has negative eigenvalues $\lambda_1 = -\mu,\ \lambda_2 = -\mu - \gamma$. The other eigenvalues of Eq.(4)are determined by equation

$$(\lambda + \mu + \epsilon)(\lambda + \mu + \delta + \alpha) - \epsilon(\frac{\beta S_0}{\varphi(S_0)} + \sigma\beta V_0) = 0. \quad (5)$$

So, if$(\mu+\epsilon)(\mu+\delta+\alpha)-\epsilon(\frac{\beta S_0}{\varphi(S_0)}+\sigma\beta V_0) > 0$, namely, $R_0 < 1$, all roots of Eq.(5) have negative real parts. If$(\mu + \epsilon)(\mu + \delta + \alpha) - \epsilon(\frac{\beta S_0}{\varphi(S_0)} + \sigma\beta V_0) = 0$,namely, $R_0 = 1$, one root of Eq.(5) is 0 and it is simple. if $R_0 > 1$, one of roots of Eq.(5) has positive real parts. Thus we have

Lemma 3.1 *If $R_0 < 1$, the disease-free equilibrium P_0 is locally stable; If $R_0 = 1$,P_0 is stable; If $R_0 > 1$,P_0 is unstable.*

Next, we prove that the disease-free equilibrium P_0 is globally asymptotically stable if $R_0 < 1$.

To obtain the global attraction of the disease-free equilibrium P_0, we need the following lemma.

Lemma 3.2 [17] *f is a bounded real-valued function in $[0, \infty)$, Letting*

$$f_\infty = \lim_{t\to+\infty} \inf f(t), \quad f^\infty = \lim_{t\to+\infty} \sup f(t).$$

where,

$$\inf f(t) = \inf\{f(u) : u \in [t, +\infty), t > 0\}, \quad \sup f(t) = \sup\{f(u) : u \in [t, +\infty), t > 0\}.$$

Assume that $f : [0, \infty) \to R$ be twice differentiable with bounded second derivative. Letting $k \to \infty$, $t_k \to \infty$ and $f(t_k)$ converges to f^∞ or f_∞, then $\lim_{k\to+\infty} f'(t_k) = 0$.

Theorem 3.3 *If $R_0 < 1$, then the disease-free equilibrium P_0 of model* (2) *is globally asymptotically stable.*

Proof. From the above discussion, we have obtained that the disease-free equilibria P_0 is locally stable as $R_0 < 1$. Next, we discussed that P_0 is globally attractive.

From the second equation of model (2), we obtain

$$\frac{dV}{dt} \le pA - (\mu + \gamma)V.$$

Let $\frac{dX}{dt} = pA - (\mu+\gamma)X$, so a solution of the equation $\frac{dX}{dt} = pA - (\mu+\gamma)X$ is a supper solution of $V(t)$. That is, $X(t) \ge V(t)$ for all $t \ge 0$.

Noting that, $X(t) \to \frac{pA}{\mu+\gamma}$ as $t \to \infty$, it follows that for a given $\epsilon_1 > 0$, there is a t_0, such that,

$$V(t) \leq X(t) \leq \frac{pA}{\mu+\gamma} + \epsilon_1, \quad for \quad t \geq t_0.$$

Consequently, $V^\infty \leq \frac{pA}{\mu+\gamma} + \epsilon_1$. Letting $\epsilon_1 \to 0$, we have $V^\infty \leq \frac{pA}{\mu+\gamma}$.

From the first equation of the model (2), we obtain

$$\frac{dS}{dt} \leq (1-p)A + \gamma(\frac{pA}{\mu+\gamma} + \epsilon_1) - \mu S.$$

Let $\frac{dY}{dt} = (1-p)A + \gamma(\frac{pA}{\mu+\gamma} + \epsilon_1) - \mu Y$, a solution of the equation $\frac{dY}{dt} = (1-p)A + \gamma(\frac{pA}{\mu+\gamma} + \epsilon_1) - \mu Y$ is a supper solution of $S(t)$. That is, $Y(t) \geq S(t), t \geq 0$.

Noting that, when $t \to \infty$, $Y(t) \to \frac{(1-p)A(\mu+\gamma)+Ap\gamma+\gamma\epsilon_1(\mu+\gamma)}{\mu(\mu+\gamma)}$. It follows that for a given $\epsilon_2 > 0$, there is a t_0, such that

$$S(t) \leq Y(t) \leq \frac{(1-p)A(\mu+\gamma) + Ap\gamma + \gamma\epsilon_1(\mu+\gamma)}{\mu(\mu+\gamma)} + \epsilon_2, \qquad for \quad t \geq t_0.$$

So

$$S^\infty \leq \frac{(1-p)A(\mu+\gamma) + Ap\gamma + \gamma\epsilon_1(\mu+\gamma)}{\mu(\mu+\gamma)} + \epsilon_2.$$

Let $\epsilon_1 \to 0, \epsilon_2 \to 0$, we have

$$S^\infty \leq \frac{(1-p)A(\mu+\gamma) + Ap\gamma}{\mu(\mu+\gamma)} = \frac{A}{\mu} - \frac{pA}{\mu+\gamma}.$$

From the forth equation of model (2), we obtain

$$I^\infty = \frac{\epsilon}{\mu+\alpha+\delta} \lim_{t\to+\infty} E(t) \leq \frac{\epsilon}{\mu+\alpha+\delta} E^\infty.$$

From the third equation of model (2), we obtain

$$E^\infty = \frac{\beta}{\mu+\epsilon} \lim_{t\to+\infty} (\frac{S(t)I(t)}{\varphi(S(t))} + \sigma V(t)I(t)) \leq \frac{\beta}{\mu+\epsilon}[\frac{\frac{A}{\mu} - \frac{pA}{\mu+\gamma}}{\varphi(\frac{A}{\mu} - \frac{pA}{\mu+\gamma})} + \sigma\frac{pA}{\mu+\gamma}]I^\infty.$$

So

$$I^\infty \leq \frac{\epsilon\beta}{(\mu+\epsilon)(\mu+\alpha+\delta)}[\frac{\frac{A}{\mu} - \frac{pA}{\mu+\gamma}}{\varphi(\frac{A}{\mu} - \frac{pA}{\mu+\gamma})} + \sigma\frac{pA}{\mu+\gamma}]I^\infty = R_0 I^\infty.$$

$$E^\infty \leq \frac{\epsilon\beta}{(\mu+\epsilon)(\mu+\alpha+\delta)}[\frac{\frac{A}{\mu} - \frac{pA}{\mu+\gamma}}{\varphi(\frac{A}{\mu} - \frac{pA}{\mu+\gamma})} + \sigma\frac{pA}{\mu+\gamma}]E^\infty = R_0 E^\infty.$$

If $R_0 < 1$, then $I^\infty \le 0$, $E^\infty \le 0$. Since $I_\infty \ge 0$, $E_\infty \ge 0$, we have, $I^\infty = I_\infty = 0$, $E^\infty = E_\infty = 0$. Thus, $t \to \infty$, $(E(t), I(t)) \to (0, 0)$.

Now, we prove the following formulas are true.

$$\lim_{t\to+\infty} S(t) = \frac{A}{\mu} - \frac{pA}{\mu+\gamma}, \quad \lim_{t\to+\infty} V(t) = \frac{pA}{\mu+\gamma}.$$

According to Lemma 3.2, we choose some sequences, $t_n \to \infty$, $s_n \to \infty$, $h_n \to \infty$, $v_n \to \infty$, such that,

$$V(v_n) \to V^\infty, \quad V(h_n) \to V_\infty, \quad S(s_n) \to S^\infty, \quad S(t_n) \to S_\infty.$$

We have $S'(s_n) \to 0$, $S'(t_n) \to 0$, $V'(v_n) \to 0$, $V'(h_n) \to 0$. Since $(E(t), I(t)) \to (0, 0)$ as $t \to \infty$.

From the second equation of the model (2), we obtain

$$pA - (\mu+\gamma) \lim_{t\to+\infty} \sup V(t) = 0, \quad pA - (\mu+\gamma) \lim_{t\to+\infty} \inf V(t) = 0.$$

Thus, $\lim_{t\to+\infty} V(t) = \frac{pA}{\mu+\gamma}$.

From the first equation of model (2), we obtain

$$(1-p)A - \mu \lim_{t\to+\infty} \sup S(t) + \frac{\gamma pA}{\mu+\gamma} = 0, \quad (1-p)A - \mu \lim_{t\to+\infty} \inf S(t) + \frac{\gamma pA}{\mu+\gamma} = 0.$$

Thus, $\lim_{t\to+\infty} S(t) = \frac{A}{\mu} - \frac{pA}{\mu+\gamma}$.

That is to say, $R_0 < 1$, the disease-free equilibrium P_0 is globally asymptotically stable.

Now, we investigate the local stability of the endemic equilibria $P^*(S^*, V^*, E^*, I^*)$.

The Jacobian matrix of model (2) at the endemic equilibria P^* as follows

$$\begin{pmatrix} -\mu - \beta I^*(\frac{S^*}{\varphi(S^*)})' & \gamma & 0 & -\frac{\beta S^*}{\varphi(S^*)} \\ 0 & -\sigma\beta I^* - \mu - \gamma & 0 & -\sigma\beta V^* \\ \beta I^*(\frac{S^*}{\varphi(S^*)})' & \sigma\beta I^* & -\mu-\epsilon & \frac{\beta S^*}{\varphi(S^*)} + \sigma\beta V^* \\ 0 & 0 & \epsilon & -(\mu+\delta+\alpha) \end{pmatrix}$$

So the corresponding characteristic equation can be found as

$$\lambda^4 + Q_1\lambda^3 + Q_2\lambda^2 + Q_3\lambda + Q_4 = 0. \tag{6}$$

Where, $Q_1 = 4\mu + \gamma + \epsilon + \delta + \alpha + \sigma\beta I^* + \beta I^*(\frac{S^*}{\varphi(S^*)})' > 0$,

$$\begin{aligned} Q_2 = {} & (\mu+\epsilon)(\mu+\delta+\alpha) + (\sigma\beta I^* + \mu + \gamma)(2\mu + \epsilon + \delta + \alpha) \\ & + (\mu + \beta I^*(\tfrac{S^*}{\varphi(S^*)})')(\sigma\beta I^* + 3\mu + \gamma + \epsilon + \delta + \alpha) > 0, \\ Q_3 = {} & (\sigma\beta I^* + \mu + \gamma)(\mu+\epsilon)(\mu+\delta+\alpha) + (\mu + \beta I^*(\tfrac{S^*}{\varphi(S^*)})')[(\mu+\epsilon)(\mu+\delta+\alpha) \\ & + (\sigma\beta I^* + \mu + \gamma)(2\mu + \epsilon + \delta + \alpha)] + \beta\epsilon\mu\tfrac{S^*}{\varphi(S^*)} > 0, \end{aligned}$$

$$Q_4 = (\mu+\beta I^*(\frac{S^*}{\varphi(S^*)})')(\sigma\beta I^*+\mu+\gamma)(\mu+\epsilon)(\mu+\delta+\alpha) +\gamma\epsilon\mu\sigma\beta V^* + \beta\epsilon\mu\frac{S^*}{\varphi(S^*)}(\sigma\beta I^*+\mu+\gamma) > 0,$$

$$H_1 = Q_1 > 0, \qquad H_2 = Q_1Q_2 - Q_3 > 0,$$

$$H_3 = \begin{vmatrix} Q_1 & Q_3 & 0 \\ 1 & Q_2 & Q_4 \\ 0 & Q_1 & Q_3 \end{vmatrix} = -Q_3^2 + Q_1Q_2Q_3 - Q_1^2Q_4 = Q_3H_2 - Q_1^2Q_4 > 0,$$

$$H_4 = Q_4H_3 > 0.$$

By the Routh-Hurwitz theorem, it follows that all the roots of the equation (6) have negative real parts. Hence, the endemic equilibrium $P^*(S^*, V^*, E^*, I^*)$ is locally asymptotically stable.

From the above discussion,we can summarize the following conclusion.

Theorem 3.4 *If $R_0 > 1$, then system (2) has a unique endemic equilibrium $P^*(S^*, V^*, E^*, I^*)$, which is locally asymptotically stable.*

4 persistence of the system (2)

In this section, we shall apply Theorem 4.6 in[17] to study the persistence of disease.

Theorem 4.1 *If $R_0 > 1$, model* (2) *is uniformly permanent.*

Proof. Let $X = \{(S, V, E, I) \mid S, V, E, I \geq 0\}$ be a metric space and $\Phi_t(S_0, V_0, E_0, I_0)$ be the solution semiflow of system (2) with $S(0) = S_0, V(0) = V_0, E(0) = E_0, I(0) = I_0$, it is easy to prove that Φ is continuous. In the previous section, we have shown that $\Omega = \{(S, V, E, I) \mid S, V, E, I \leq \frac{A}{\mu}\}$ is a closed and positively invariant set of X , and the metric space is a compact one.

Thus, there exists a compact set N, in which all solutions of system (2) initiated from Ω ultimately enter and remain in it forever. Let $\omega(y)$ be the ω-limit set of the solution of system (2) starting from Ω. We need to show the following set holds

$$N_\alpha = \bigcup \omega(y)_{y\in Y}, \qquad Y = \{x_0 \in \partial\Omega \mid x(t, x_0) \in \partial\Omega, \quad \forall t > 0\}.$$

There always exists the unique disease-free equilibrium $P_0(S_0, V_0, 0, 0)$ on the boundary of Ω, from the previous proof we know that, P_0 is unstable as $R_0 > 1$, P_0 is the unique largest invariant subset on the boundary of Ω, and P_0 is a covering of Ω, which is isolated and acyclic. Thus, $N_\alpha = \{P_0\}$. Since that the closed positive octant is positively invariant for system (2), it follows that

$$\lim_{t\to+\infty} \sup d(x(t, x_0), P_0) = \lim_{t\to+\infty} \inf d(x(t, x_0), P_0) = 0.$$

To show model (2) is uniformly permanent, according to Lemma 4 of [15], we only need to verify $W^+(P_0)\bigcap\dot{\Omega}=\emptyset$.

where $W^+(P_0)$ denotes the stable manifold of P_0. If $R_0>1$, P_0 is unstable. In particular, the Jacobian matrix of system (2) has one eigenvalue with positive real part, which denotes as λ_+, And three eigenvalues with negative real part, which we respectively, denote as λ_-, $-\mu$, and $-(\mu+\lambda)$. (λ_- may be equal to $-\mu$ or $-(\mu+\lambda)$). We shall proceed by determining the location of $E(P_0)$(the stable eigenspace of P_0). Then the eigenvector associated to λ_- is $(0,0,n_1,n_2)^T$ as $\lambda_-\neq-\mu$ and $\lambda_-\neq-(\mu+\lambda)$, where, n_1,n_2 satisfy the eigenvector equation

$$\begin{pmatrix} -(\mu+\epsilon) & \frac{\beta(\frac{A}{\mu}-\frac{pA}{\mu+\gamma})}{\varphi(\frac{A}{\mu}-\frac{pA}{\mu+\gamma})}+\sigma\beta\frac{pA}{\mu+\gamma} \\ \epsilon & -(\mu+\delta+\alpha) \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}=\lambda_-\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}. \tag{7}$$

In the rest of the proof, if we show that in both cases

(1)$\lambda_-=-(\mu+\omega)$ or $\lambda_-=-\mu$,

(2)$\lambda_-\neq-(\mu+\omega)$ and $\lambda_-\neq-\mu$,

the vector $(n_1,n_2)^T\overline{\in}R^2_+$, then the proof of Theorem 4.1 is complete.

In fact, by the definition of an irreducible matrix, the matrix in (7) is an irreducible Metzler matrix, we note M. Thus, $M+NI_{2\times2}$ is a nonnegative irreducible matrix, where, N is a sufficiently large positive constant, $I_{2\times2}$ is the identify matrix. Thus the conditions of the Perron-Frobenius theorem in [17] are satisfied. By the PerronCFrobenius theorem, we know that M possesses the dominant eigenvalue λ_+. But the Perron-Frobenius theorem also implies that every eigenvector does not belong to the closed positive octant since it is not associated with the dominant eigenvalue. This means that $(n_1,n_2)^T\overline{\in}R^2_+$. Therefore, $E^+(P_0)\bigcap\dot{\Omega}=\emptyset$. Namely, $W^+(P_0)\bigcap\dot{\Omega}=\emptyset$. Thus, if $R_0>1$, model (2) is uniformly permanent.

5 Global stability of the endemic equilibrium

In this section, we apply the geometrical approach[2] to investigate the global stability of the endemic equilibrium P^* in the feasible region Ω.

Lemma 5.1 [2] *consider the differential equation*

$$x'=f(x). \tag{8}$$

and its corresponding periodic linear system

$$z'=\frac{\partial f^{[2]}}{\partial x}(p(t))z(t). \tag{9}$$

Where, $\frac{\partial f^{[2]}}{\partial x}$ *is the second additive compound matrix of* $\frac{\partial f}{\partial x}$ *and* $\Theta = \{p(t) : 0 \leq t \leq \omega\}$ *is the periodic orbit of* (8).

We make the following four assumptions

(1) *there is a compact absorbing set* $K \subset D$ *and a unique equilibrium* $\overline{x} \in D$.

(2) *model* (8) *satisfies the Poincaré − Bendixson property.*

(3) (9) *is asymptotically stable for each periodic solution* $x = p(t)$ *to* (8) *with* $p(0) \in D$

(4) $(-1)^n det(\frac{\partial f}{\partial x}(\overline{x})) > 0$.

Then, the unique equilibrium $\overline{x}$ *of model* (8) *is globally asymptotically stable in* D.

Theorem 5.2 *If* $R_0 > 1$, *the unique positive equilibrium* P^* *of model* (2) *is globally asymptotically stable in* Ω.

Proof. we only need to prove that four assumptions of Lemma 5.1 hold.

If $R_0 > 1$, model (2) is uniformly permanent, and the unique positive equilibrium P^* of model (2) is locally asymptotically stable in Ω. So there is a compact absorbing set $K \subset \Omega$. Assumption (1) holds.

The Jacobian matrix of model (2) is as follow

$$J(P) = \begin{pmatrix} -\mu - \beta I(\frac{S}{\varphi(S)})' & \gamma & 0 & -\frac{\beta S}{\varphi(S)} \\ 0 & -\sigma\beta I - \mu - \gamma & 0 & -\sigma\beta V \\ -\mu & -\mu & -\mu - \epsilon & 0 \\ 0 & 0 & \epsilon & -(\mu + \delta + \alpha) \end{pmatrix}.$$

Choosing the matrix H as $H = diag(-1, -1, 1, -1)$, it is easy to prove that HJH has non-positive off-diagonal elements, so we can see that system (2) is competitive. This verifies the assumption (2).

The second additive compound matrix of the matrix J is

$$J^{[2]} = \begin{pmatrix} \Phi_1 & 0 & -\sigma\beta V & 0 & \frac{\beta S}{\varphi(S)} & 0 \\ -\mu & \Phi_2 & 0 & \gamma & 0 & \frac{\beta S}{\varphi(S)} \\ 0 & \epsilon & \Phi_3 & 0 & \gamma & 0 \\ \mu & 0 & 0 & \Phi_4 & 0 & \sigma\beta V \\ 0 & 0 & 0 & \epsilon & \Phi_5 & 0 \\ 0 & 0 & -\mu & 0 & -\mu & \Phi_6 \end{pmatrix},$$

where,

$\Phi_1 = -(2\mu + \sigma\beta I + \gamma + \beta I(\frac{S}{\varphi(S)})')$, $\Phi_2 = -2\mu - \epsilon - \beta I(\frac{S}{\varphi(S)})'$,
$\Phi_3 = -2\mu - \delta - \alpha - \beta I(\frac{S}{\varphi(S)})'$, $\Phi_4 = -\sigma\beta I - 2\mu - \gamma - \epsilon$,
$\Phi_5 = -\sigma\beta I - 2\mu - \gamma - \delta - \alpha$, $\Phi_6 = -2\mu - \epsilon - \delta - \alpha$.

We have the second compound system of the model (2) in a periodic solution

$$\begin{cases} \frac{dX}{dt} = -(2\mu+\sigma\beta I+\gamma+\beta I(\frac{S}{\varphi(S)})')X - \sigma\beta VZ + \frac{\beta S}{\varphi(S)}M, \\ \frac{dY}{dt} = -\mu X - (2\mu+\epsilon+\beta I(\frac{S}{\varphi(S)})')Y + \gamma L + \frac{\beta S}{\varphi(S)}N, \\ \frac{dZ}{dt} = \epsilon Y - (2\mu+\delta+\alpha+\beta I(\frac{S}{\varphi(S)})')Z + \gamma M, \\ \frac{dL}{dt} = \mu X - (\sigma\beta I + 2\mu+\gamma+\epsilon)L + \sigma\beta VN, \\ \frac{dM}{dt} = \epsilon L - (\sigma\beta I + 2\mu + \gamma+\delta+\alpha)M, \\ \frac{dN}{dt} = -\mu Z - \mu M - (2\mu+\epsilon+\delta+\alpha)N. \end{cases} \tag{10}$$

Next, we prove that system (10) is asymptotically stable.We can choose Liapunov function as

$$V(X,Y,Z,L,M,N;S,V,E,I) = \sup\{|X|+|Y|+|L|,\ \frac{E}{I}|Z|+|M|+|N|\}.$$

By the uniform persistence, we obtain that the orbit of $P(t) = (S(t), V(t), E(t), I(t))$ remains a positive distance from the boundary of Ω, therefore we can know there exists a constant $c_1 > 0$, such that,

$$V(X,Y,Z,L,M,N;S,V,E,I) \geq c_1 \sup\{|X|,|Y|,|Z|,|L|,|M|,|N|\},$$

for all $(X,Y,Z,L,M,N) \in R^6$ and $(S,V,E,I) \in P(t)$.

Direct calculations lead to the following differential inequalities. Noting that,

$$\begin{aligned} D_+|X(t)| \leq\ & -(2\mu+\sigma\beta I+\gamma+\beta I(\tfrac{S}{\varphi(S)})')|X(t)| + \tfrac{\beta S}{\varphi(S)}|M(t)| \\ \leq\ & -(2\mu+\epsilon)|X(t)| + \tfrac{\beta S}{\varphi(S)}(|M(t)|+|Z(t)|+|N(t)|), \quad (\gamma \geq \epsilon), \\ D_+|Y(t)| \leq\ & -(2\mu+\epsilon)|Y(t)| + \gamma|L(t)| + \tfrac{\beta S}{\varphi(S)}(|M(t)|+|Z(t)|+|N(t)|), \\ D_+|Z(t)| \leq\ & \epsilon|Y(t)| - (2\mu+\delta+\alpha)|Z(t)| + \gamma|M(t)|, \\ D_+|L(t)| \leq\ & \mu|X(t)| - (2\mu+\epsilon)|L(t)| + \sigma\beta V|N(t)|, \\ D_+|M(t)| \leq\ & \epsilon|L(t)| - (2\mu+\delta+\alpha)|M(t)| - \gamma|M(t)|, \\ D_+|N(t)| \leq\ & -(2\mu+\delta+\alpha)|N(t)|. \end{aligned}$$

So,

$$\begin{aligned} D_+(|X|+|Y|+|L|) \leq\ & -(2\mu+\epsilon)(|X|+|Y|+|L|) + (\tfrac{\beta S}{\varphi(S)} + \sigma\beta V)(|M|+|Z|+|N|) \\ =\ & -(2\mu+\epsilon)(|X|+|Y|+|L|) + \tfrac{E}{I}(\tfrac{\beta SI}{E\varphi(S)} + \sigma\beta V\tfrac{I}{E})(|M|+|Z|+|N|), \end{aligned}$$

$$D_+(|Z|+|M|+|N|) \leq -(2\mu+\delta+\alpha)(|Z|+|M|+|N|) + \epsilon(|X|+|Y|+|L|).$$

Then,

$$D_+\frac{E}{I}(|Z|+|M|+|N|) \leq \frac{E}{I}\epsilon(|X|+|Y|+|L|) + (\frac{E'}{E} - \frac{I'}{I} - 2\mu - \delta - \alpha)\frac{E}{I}(|Z|+|M|+|N|).$$

From the previous formula, we lead to

$$D_+|V(t)| \le \max\{g_1(t), g_2(t)\}V(t),$$

where,

$$g_1(t) = -2\mu - \epsilon + \frac{\beta SI}{E\varphi(S)} + \sigma\beta V\frac{I}{E},\ g_2(t) = \epsilon\frac{E}{I} + \frac{E'}{E} - \frac{I'}{I} - 2\mu - \delta - \alpha.$$

From the model (2) we obtain

$$\frac{E'}{E} = \frac{\beta SI}{E\varphi(S)} + \sigma\beta V\frac{I}{E} - \mu - \epsilon,\quad \frac{I'}{I} = \epsilon\frac{E}{I} - (\mu + \delta + \alpha).$$

so,

$$g_1(t) = \frac{E'}{E} - \mu,\quad g_2(t) = \frac{E'}{E} - \mu.$$

Then,

$$\int_0^\omega \max\{g_1(t), g_2(t)\}dt = \ln E(t)\ |_0^\omega - \omega\mu = -\omega\mu.$$

$$D_+|V(t)| \le \max\{g_1(t), g_2(t)\}V(t).$$

which implies that $(X(t), Y(t), Z(t), L(t), M(t), N(t)) \to 0$, as $t \to \infty$. As a result, the second compound system (10) is asymptotically stable.

This verifies the assumption (3) of Lemma 5.1.

Let $J(P^*)$ be the *Jacobian* matrix of the model (2) at P^*, we have

$$\begin{aligned}
\det(J(P^*)) &= \begin{vmatrix} -\mu - \beta I^*(\frac{S^*}{\varphi(S^*)})' & \gamma & 0 & -\frac{\beta S^*}{\varphi(S^*)} \\ 0 & -\sigma\beta I^* - \mu - \gamma & 0 & -\sigma\beta V^* \\ -\mu & -\mu & -\mu - \epsilon & 0 \\ 0 & 0 & \epsilon & -(\mu + \delta + \alpha) \end{vmatrix} \\
&= -\epsilon \begin{vmatrix} -\mu - \beta I^*(\frac{S^*}{\varphi(S^*)})' & \gamma & -\frac{\beta S^*}{\varphi(S^*)} \\ 0 & -\sigma\beta I^* - \mu - \gamma & -\sigma\beta V^* \\ -\mu & -\mu & 0 \end{vmatrix} \\
&\quad + (\mu + \delta + \alpha)(\mu + \epsilon) \begin{vmatrix} -\mu - \beta I^*(\frac{S^*}{\varphi(S^*)})' & \gamma \\ 0 & -\sigma\beta I^* - \mu - \gamma \end{vmatrix} \\
&= -\epsilon\mu[(\gamma + \mu + \beta I^*(\frac{S^*}{\varphi(S^*)})')(-\sigma\beta V^*) - (\sigma\beta I^* + \mu + \gamma)\frac{\beta S^*}{\varphi(S^*)}] \\
&\quad + (\mu + \delta + \alpha)(\mu + \epsilon)(\mu + \beta I^*(\frac{S^*}{\varphi(S^*)})')(\sigma\beta I^* + \mu + \gamma) > 0.
\end{aligned}$$

Thus, $(-1)^6 \det(J(P^*)) > 0$. The assumption (4) holds.

This verifies all the assumptions of Lemma 5.1 , so P^* is globally asymptotically stable in Ω.

6 Concluding remarks

In this paper, we propose an SVEIR model with a nonlinear incidence rate and vaccination. We investigate the global dynamics behavior of the reduced system, For the model (2), we obtain the basic reproduction number $R_0 = \frac{\epsilon\beta(\frac{\frac{A}{\mu}-\frac{pA}{\mu+\gamma}}{\varphi(\frac{A}{\mu}-\frac{pA}{\mu+\gamma})}+\sigma\frac{pA}{\mu+\gamma})}{(\mu+\epsilon)(\mu+\delta+\alpha)}$, then we obtained that the disease eradicate from the community if $R_0 \leq 1$; the disease will persist if $R_0 > 1$. That is to say, it is necessary and important for public health management to control an epidemic by increasing the duration of the loss of immunity induced by vaccination to decrease the basic reproduction number R_0, until less then unit,which will lead to the disease eradication.

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