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Implicit and explicit iterative schemes for variational inequalities and fixed point problems of a countable family of strict pseudo-contractions

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Abstract

Let H be a Hilbert space and let $\{T_i\}_{i=1}^{\infty}$ be a countable family of strict pseudo-contractions of H into itself such that $C = \cap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Assume that F is a nonlinear operator which is κ-Lipschitzian and η-strong monotone on C. In this paper, we devise an implicit and an explicit iterative schemes $\{x_n\}$ from an arbitrary initial point $x_0 \in H$ for the countable family of mappings $\{T_i\}_{i=1}^{\infty}$ and prove that $\{x_n\}$ converges strongly to the solution x^* of the variational inequality

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

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1 Introduction

Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $F : H \to H$ be a nonlinear operator. The variational inequality problem on F is to find a point $x^* \in C$ such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C.$$

The variational inequality problem is denoted by $VI(F, C)$ [1].

F is called to be κ-Lischitzian and η-strongly monotone if there exist constants $\kappa, \eta > 0$ such that

$$\|Fx - Fy\| \leq \kappa \|x - y\|, \quad \text{for all } x, y \in H,$$

and

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \quad \text{for all } x, y \in H,$$

respectively.

It is well known that if F is strongly monotone and Lipschitzian on C, then $VI(F, C)$ has a unique solution. An important problem is how to find a solution of $VI(F, C)$.

It is known that the $VI(F, C)$ is equivalent to the fixed point equation [2]

$$u^* = P_C(u^* - \mu F(u^*)), \tag{1.1}$$

where P_C is the projection from H onto C; i.e.,

$$P_C x = \min_{y \in C} \|x - y\|, \quad \forall x \in H,$$

and where $\mu > 0$ is an arbitrarily fixed constant. So, the fixed point methods can be implemented to find a solution of the $VI(F, C)$.

The fixed point formulation (1.1) involves the projection P_C, which may not be easy to compute, due to the complexity of the convex set C. Hence, in order to reduce the complexity probably caused by the projection P_C, Yamada [2] introduced the hybrid steepest-descent method for solving the $VI(F, C)$ by replacing P_C with a nonexpansive mapping T. Recall that a mapping $T : H \to H$ is called nonexpansive if for all $x, y \in H$, one holds

$$\|Tx - Ty\| \leq \|x - y\|.$$

The set of fixed points of T is denoted by $Fix(T)$. More precisely, Yamada [2] gave the following iterative scheme:

$$u_0 \in H, \quad u_{n+1} = Tu_n - \lambda_{n+1} \mu F(Tu_n), \quad n \geq 0, \tag{1.2}$$

where $T : H \to H$ is a nonexpansive mapping, F is η-strongly monotone and κ-Lipschitzian on $C = Fix(T)$, $\{\lambda_n\}$ is a sequence in $(0, 1)$ and μ is a fixed number with $0 < \mu < 2\eta/\kappa^2$. Yamada [2] proved that if the sequence $\{\lambda_n\}$ satisfies the conditions:

(L1) $\lim_{n\to\infty} \lambda_n = 0$,
(L2) $\sum_{n=1}^{\infty} \lambda_n = \infty$,
(L3) $\lim_{n\to\infty} (\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$,

then $\{u_n\}$ generated by (1.2) converges strongly to the unique solution of $VI(F, C)$.

Let $N \geq 1$ be an integer and $\{T_i\}_{i=1}^N : H \to H$ be a finite family of nonexpansive mappings such that $C = \cap_{i=1}^N Fix(T_i) = Fix(T_1 T_2 \cdots T_N) =$

$Fix(T_N T_1 \cdots T_{N-1}) = \cdots = Fix(T_2 T_3 \cdots T_N T_1)$. Yamada' another algorithm [2] is as follows:

$$u_0 \in H, \quad u_{n+1} = T_{[n+1]}u_n - \lambda_{n+1}\mu F(T_{[n+1]}u_n), \quad n \geq 0, \tag{1.3}$$

where $T_{[n+1]} = T_{n+1 \mod N}$, for $n \geq 0$, $\mu \in (0, 2\eta/\kappa^2)$ and $\{\lambda_n\}$ is a sequence in $(0,1)$ satisfying the conditions (L1), (L2) and (L4),

(L4) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}|$ is convergent.

Under these conditions, Yamada [2] proved that $\{u_n\}$ generated by (1.3) converges strongly to the unique solution of $VI(F, C)$.

Since Yamada's hybrid steepest-descent method for solving variational inequalities [2], there are much research on this aspect; see, e.g., [3–7, 10, 11].

Recently, Zeng and Yao [8] introduced the following implicit method:

$$\begin{aligned}
&x_1 = \beta_1 x_0 + (1-\beta_1)[T_1 x_1 - \lambda_1 \mu F(T_1 x_1)],\\
&x_2 = \beta_1 x_1 + (1-\beta_1)[T_2 x_2 - \lambda_2 \mu F(T_2 x_2)],\\
&\vdots\\
&x_N = \beta_N x_{N-1} + (1-\beta_N)[T_N x_N - \lambda_N \mu F(T_N x_N)],\\
&x_{N+1} = \beta_{N+1} x_N + (1-\beta_{N+1})[T_1 x_{N+1} - \lambda_{N+1} \mu F(T_1 x_{N+1})],
\end{aligned}$$

The scheme is written in a compact form as

$$x_n = \beta_n x_{n-1} + (1-\beta_n)[T_[n]x_n - \lambda_n \mu F(T_[n]x_n)], \quad n \geq 1 \tag{1.4}$$

They proved the following result.

Theorem 1.1 ([8]). *Let H be a real Hilbert space and $F : H \to H$ a mapping such that for some constants $L, \eta > 0$, F is L-Lipschitz continuous and η-strongly monotone. Let $\{T_i\}_{i=1}^N$ be N nonexpansive self-mappings of H such that $C = \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$. Let $\mu \in (0, 2\eta/L^2)$, and let $x_0 \in H$, with $\{\lambda_n\}_{n=1}^{\infty} \subset [0,1)$ and $\{\beta_n\}_{n=1}^{\infty} \subset (0,1)$ satisfying the conditions: $\sum_{i=1}^{\infty} \lambda_n < \infty$ and $\alpha \leq \beta_n \leq \beta$, $n \geq 1$, for some $\alpha, \beta \in (0,1)$. Then, the sequence $\{x_n\}$ defined by* (1.4) *converges weakly to a common fixed point of the mapping $\{T_i\}_{i=1}^N$. The convergence is strong if and only if* $\liminf_{n\to\infty} d(x_n, C) = 0$.

As we know, in general it is hard to get the strong convergence result for implicit iterative scheme. However, in this paper, we introduce an implicit iterative schemes with a countable family of strict pseudo-contractions for solving a variable inequality in a Hilbert space and prove the strong convergence for the implicit iteration. The control parameters for the implicit iteration is simple. Besides the implicit iteration, we also design an explicit iteration for the family of strict pseudo-contractions and prove a strong convergence result.

2 Preliminaries

Let H be a Hilbert space and let T be a nonexpansive mapping of H into itself such that $Fix(T) \neq \emptyset$. For all $\hat{x} \in Fix(T)$ and all $x \in H$, we have

$$\begin{aligned} \|x-\hat{x}\|^2 &\geq \|Tx-T\hat{x}\|^2 = \|Tx-\hat{x}\|^2 = \|Tx-x+(x-\hat{x})\|^2 \\ &= \|Tx-x\|^2 + \|x-\hat{x}\|^2 + 2\langle Tx-x, x-\hat{x}\rangle \end{aligned}$$

and hence

$$\|Tx-x\|^2 \leq 2\langle x-T^n x, x-\hat{x}\rangle \quad \forall \hat{x} \in Fix(T), \forall x \in H. \tag{2.1}$$

Let $T : H \to H$ be a nonexpansive mapping and let $F : H \to H$ be a κ-Lipschitzian and η-strong monotone nonlinear operator. The following lemma may be found in [2].

Lemma 2.1 ([2]). *Assume $\lambda \in (0,1)$ and $\mu \in (0, 2\eta/\kappa^2)$. Define a mapping $T^\lambda : H \to H$ by*

$$T^\lambda x = Tx - \lambda\mu F(Tx) \quad \forall x \in H.$$

Then $\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\tau)\|x-y\|$ for all $x, y \in H$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1)$.

Obviously, if $T = I$, then $T^\lambda = I - \lambda\mu F$ and by Lemma 2.1 we have

$$\|T^\lambda x - T^\lambda y\| = \|(I-\lambda\mu F)x - (I-\lambda\mu F)y\| \leq (1-\lambda\tau)\|x-y\| \tag{2.2}$$

for all $x, y \in H$.

Lemma 2.2 ([9, Lemma 3.1]). *Let $\{s_n\}$, $\{c_n\}$ be the sequences of nonnegative real numbers and let $\{a_n\} \subset (0,1)$. Suppose $\{b_n\}$ is a real number sequence such that*

$$s_{n+1} \leq (1-a_n)s_n + b_n + c_n, \quad n \geq 0.$$

Assume $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

(1) *If $b_n \leq \beta a_n$ where $(\beta \geq 0)$, then $\{s_n\}$ is a bounded sequence.*

(2) *If we have*

$$\sum_{n=0}^{\infty} a_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \frac{b_n}{a_n} \leq 0,$$

then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3. *Let H be a real Hilbert space. There holds the following identity:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$$

for all $x, y \in H$.

3. Main results

The following are the main results of this paper.

Theorem 3.1. *Let H be a real Hilbert space and $\{T_i\}_{i=1}^{\infty} : H \to H$ be a countable family of κ_i-strict pseudo-contractions such that $C = \bigcap_{i=1}^{\infty} Fix(T_i) \neq \emptyset$. Let $F : H \to H$ be a nonlinear mapping which is κ-Lipschitz continuous and η-strongly monotone on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and set $\alpha_0 = 1$. Assume the sequences $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,1)$ satisfy the following conditions:*

(i) $\{\alpha_n\}$ *is strictly decreasing and* $\lim_{n\to\infty} \alpha_n = 0$*;*
(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Then the implicit iterative scheme $\{x_n\}$ generated by the following manner:

$$x_0 \in H, \; x_n = \alpha_n T^{\lambda_n} x_{n-1} + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) S_i x_n, \quad n \geq 1, \tag{3.1}$$

where $T^{\lambda_n} x_{n-1} = x_{n-1} - \lambda_n \mu F(x_{n-1})$ and $S_i = \kappa_i I + (1-\kappa_i) T_i$ for each $i \geq 1$, converges strongly to the solution x^* of $VI(F,C)$, i.e., x^* solves the following variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{3.2}$$

Proof. We first shows that $\{x_n\}$ generated by (3.1) is well defined. Fixed a point $u \in H$, by (2.2) we have(note that $\{\alpha_n\}$ is strictly decreasing)

$$\begin{aligned}
&\|\alpha_n T^{\lambda_n} u + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) S_i x - (\alpha_n T^{\lambda_n} u + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) S_i y)\| \\
&= \| \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(S_i x - S_i y)\| \\
&\leq \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|S_i x - S_i y\| \\
&\leq \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|x - y\| \\
&= (1 - \alpha_n)\|x - y\| < \|x - y\|,
\end{aligned}$$

for all $x, y \in H$. This shows that for each $n \geq 1$, there exists $x_n \in H$ satisfying (3.1). Therefore, the iterative scheme $\{x_n\}$ generated by (3.1) is well defined.

Next we prove that $\{x_n\}$ is bounded.

It follows from [12] that each S_i is nonexpansive and $Fix(T_i) = Fix(S_i)$. Hence, for each $p \in \bigcap_{i=1}^{\infty} Fix(T_i)$, one has $p \in \bigcap_{i=1}^{\infty} Fix(S_i)$ and by (3.1) we have

$$\begin{aligned}\|x_n - p\| &= \|\alpha_n(T^{\lambda_n}x_{n-1} - p) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(S_i x_n - p)\| \\ &\leq \alpha_n\|T^{\lambda_n}x_{n-1} - p\| + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|S_i x_n - p\| \\ &\leq \alpha_n\|T^{\lambda_n}x_{n-1} - T^{\lambda_n}p\| + \alpha_n\|T^{\lambda_n}p - p\| + (1 - \alpha_n)\|x_n - p\| \\ &\leq \alpha_n(1 - \lambda_n\tau)\|x_{n-1} - p\| + \alpha_n\lambda_n\mu\|F(p)\| + (1 - \alpha_n)\|x_n - p\|,\end{aligned}$$

which implies that

$$\|x_n - p\| \leq (1 - \lambda_n\tau)\|x_{n-1} - p\| + \lambda_n\mu\|F(p)\| \leq \max\{\|x_{n-1} - p\|, \frac{\mu}{\tau}\|F(p)\|\}.$$

By induction, we obtain $\|x_n - p\| \leq \max\|x_0 - p\|, \frac{\mu}{\tau}\|F(p)\|$. Hence, $\{x_n\}$ is bounded and so is $\{T^{\lambda_n}x_{n-1}\}$.

Now, we prove that

$$\lim_{n\to\infty}\|x_n - S_i x_n\| = 0, \quad i \geq 1. \tag{3.3}$$

It follows from (3.1) that

$$\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(x_n - S_i x_n) + \alpha_n x_n = \alpha_n T^{\lambda_n}x_{n-1},$$

that is

$$\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)(x_n - S_i x_n) = \alpha_n(T^{\lambda_n}x_{n-1} - x_n).$$

Hence, for any $p \in \bigcap_{i=1}^{\infty} Fix(S_i)$, we get

$$\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\langle x_n - S_i x_n, x_n - p\rangle = \alpha_n\langle T^{\lambda_n}x_{n-1} - x_n, x_n - p\rangle. \tag{3.4}$$

Since each S_i is nonexpansive, by (2.1) we have

$$\|S_i x_n - x_n\|^2 \leq 2\langle x_n - S_i x_n, x_n - p\rangle.$$

Hence, combining this inequality with (3.4), we get (note that $\alpha_{i-1} - \alpha_i > 0$ for each $i \geq 1$)

$$\begin{aligned}\frac{1}{2}(\alpha_{i-1} - \alpha_i)\|S_i x_n - x_n\|^2 &\leq \frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|S_i x_n - x_n\|^2 \\ &\leq \alpha_n\langle T^{\lambda_n}x_{n-1} - x_n, x_n - p\rangle,\end{aligned}$$

that is,

$$\|S_i x_n - x_n\|^2 \le \frac{2\alpha_n}{\alpha_{i-1} - \alpha_i} \langle T^{\lambda_n} x_{n-1} - x_n, x_n - p \rangle.$$

Since $\{x_n\}$ and $\{T^{\lambda_n} x_{n-1}\}$ are both bounded, there exists a constant $M > 0$ such that

$$\|S_i x_n - x_n\|^2 \le \frac{\alpha_n M}{\alpha_{i-1} - \alpha_i}. \tag{3.5}$$

It follows from (3.5) and condition (i) that (3.3) holds.

Finally, we prove the desired result. To this end, we first prove that

$$\limsup_{n\to\infty} \langle -Fx^*, x_n - x^* \rangle \le 0. \tag{3.6}$$

To prove this, we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle -Fx^*, x_n - x^* \rangle = \lim_{i\to\infty} \langle -Fx^*, x_{n_i} - x^* \rangle.$$

Without loss of generality, we may further assume that $x_{n_i} \rightharpoonup \hat{x}$ for some $\hat{x} \in H$. From (3.2) and demiclosed principle we get $\hat{x} \in Fix(S_i)$, $i \ge 1$. Hence $\hat{x} \in C = \bigcap_{i=1}^{\infty} Fix(T_i)$. Since F is strongly monotone and Lipschitzian on C, $VI(F, C)$ is nonempty. Assume that $x^* \in C$ is a solution of $VI(F, C)$ which implies

$$\limsup_{n\to\infty} \langle -Fx^*, x_n - x^* \rangle = \langle -Fx^*, \hat{x} - x^* \rangle \le 0.$$

From Lemma 2.3 we have

$$\begin{aligned}
& \|x_n - x^*\|^2 \\
= \; & \left\| [\alpha_n (T^{\lambda_n} x_{n-1} - T^{\lambda_n} x^*) + \textstyle\sum_{i=1}^{n} (\alpha_{i-1} - \alpha_n)(S_i x_n - x^*)] + \alpha_n (T^{\lambda_n} x^* - x^*) \right\|^2 \\
\le \; & \left\| \alpha_n (T^{\lambda_n} x_{n-1} - T^{\lambda_n} x^*) + \textstyle\sum_{i=1}^{n} (\alpha_{i-1} - \alpha_n)(S_i x_n - x^*) \right\|^2 \\
& + 2\alpha_n \langle T^{\lambda_n} x^* - x^*, x_n - x^* \rangle \\
\le \; & \alpha_n \|T^{\lambda_n} x_{n-1} - T^{\lambda_n} x^*\|^2 + \textstyle\sum_{i=1}^{n} (\alpha_{i-1} - \alpha_n) \|S_i x_n - x^*\|^2 \\
& + 2\alpha_n \langle T^{\lambda_n} x^* - x^*, x_n - x^* \rangle \\
\le \; & \alpha_n (1 - \lambda_n \tau) \|x_{n-1} - x^*\|^2 + \textstyle\sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|x_n - x^*\|^2 \\
& + 2\alpha_n \langle T^{\lambda_n} x^* - x^*, x_n - x^* \rangle \\
= \; & \alpha_n (1 - \lambda_n \tau) \|x_{n-1} - x^*\|^2 + (1 - \alpha_n) \|x_n - x^*\|^2 \\
& + 2\alpha_n \lambda_n \mu \langle -F(x^*), x_n - x^* \rangle
\end{aligned}$$

which implies

$$\|x_n - x^*\|^2 \le (1 - \lambda_n \tau) \|x_{n-1} - x^*\| + 2\lambda_n \mu \langle -F(x^*), x_n - x^* \rangle$$

Since $\sum_{n=0}^{\infty} \lambda_n = \infty$ and $\limsup_{n\to\infty} \langle -F(x^*), x_n - x^* \rangle \le 0$, by Lemma 2.2(2) we conclude that $\{x_n\}$ strongly converges to x^*, which completes the proof. $\square$

If $T_i = T$ for all $i \ge 1$ in Theorem 3.1, then we have the following corollary:

Corollary 3.2. *Let H be a real Hilbert space and $T : H \to H$ be a κ'-strict pseudo-contraction such that $C = Fix(T) \neq \emptyset$. Let $F : H \to H$ be a nonlinear mapping which is κ-Lipschitz continuous and η-strongly monotone on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Assume the sequences $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,1)$ satisfy the following conditions:*

(i) $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\lim_{n\to\infty} \alpha_n = 0$*;*
(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Then the iterative scheme $\{x_n\}$ generated by the following manner:

$$x_0 \in H,\ x_n = \alpha_n\big(x_{n-1} - \lambda_n \mu F(x_{n-1})\big) + (1-\alpha_n)Sx_n, \quad n \geq 1,$$

where S is defined by $Sx = \kappa' x + (1-\kappa')Tx$ for all $x \in H$, converges strongly to the solution x^* of $VI(F,C)$, i.e., x^* solves the following variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

If each T_i is 0-strict pseudo-contraction in Theorem 3.1, i.e., each T_i is nonexpansive, then Theorem 3.1 is reduced to the following

Corollary 3.3. *Let H be a real Hilbert space and $\{T_i\} : H \to H$ be a countable family of nonexpansive mappings such that $C = Fix(T_i) \neq \emptyset$. Let $F : H \to H$ be a nonlinear mapping which is κ-Lipschitz continuous and η-strongly monotone on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and set $\alpha_0 = 1$. Assume the sequences $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,1)$ satisfy the following conditions:*

(i) $\{\alpha_n\}$ *is strictly decreasing, and* $\lim_{n\to\infty} \alpha_n = 0$*;*
(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$.

Then the iterative scheme $\{x_n\}$ generated by the following manner:

$$x_0 \in H,\ x_n = \alpha_n\big(x_{n-1} - \lambda_n \mu F(x_{n-1})\big) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i x_n, \quad n \geq 1,$$

converges strongly to the solution x^* of $VI(F,C)$, i.e., x^* solves the following variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

Next we prove a strong convergence theorem for an explicit iterative scheme as follows.

Theorem 3.4. *Let H be a real Hilbert space and $\{T_i\}_{i=1}^{\infty} : H \to H$ be a countable family of κ_i-strict pseudo-contractions such that $C = \bigcap_{i=1}^{\infty} Fix(T_i) \neq$*

$\emptyset$. Let $F : H \to H$ be a nonlinear mapping which is κ-Lipschitz continuous and η-strongly monotone on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and set $\alpha_0 = 1$. Assume the sequences $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [\lambda, 1)$ with $\lambda \in (0,1)$ satisfy the following conditions:

(i) $\{\alpha_n\}$ *is strictly decreasing,* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty}(1 - \frac{\alpha_{n-1}}{\alpha_n}) = 0$*;*
(ii) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$.

Then the iterative sequence

$$x_0 \in H, \; x_{n+1} = \alpha_n(x_n - \lambda_n \mu F(x_n)) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) S_i x_n, \quad n \geq 0, \quad (3.7)$$

where $S_i x = \kappa_i x + (1-\kappa_i) T_i x$ for all $x \in H$, converges strongly to the solution x^* of $VI(F, C)$, i.e., x^* solves the following variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

Proof. Denote $T^{\lambda_n} x_n = x_n - \lambda_n \mu F(x_n)$ for each $n \geq 1$. We first prove that $\{x_n\}$ is bounded. For any$p \in \bigcap_{i=1}^{\infty} Fix(T_i)$, one has $p \in \bigcap_{i=1}^{\infty} Fix(S_i)$ by [12]. From (3.7) we have (note that $\{\alpha_n\}$ is strictly decreasing)

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n(T^{\lambda_n} x_n - p) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i)(S_i x_n - p)\| \\
&\leq \alpha_n \|T^{\lambda_n} x_n - p\| + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) \|S_i x_{n-1} - p\| \\
&\leq \alpha_n \|T^{\lambda_n} x_n - T^{\lambda_n} p\| + \alpha_n \|T^{\lambda_n} p - p\| + (1 - \alpha_n)\|x_n - p\| \\
&\leq \alpha_n (1 - \lambda_n \tau)\|x_n - p\| + \alpha_n \lambda_n \mu \|F(p)\| + (1 - \alpha_n)\|x_n - p\| \\
&= (1 - \alpha_n \lambda_n \tau)\|x_n - p\| + \alpha_n \lambda_n \mu \|F(p)\| \\
&\leq \max\{\|x_n - p\|, \frac{\mu}{\tau} \mu \|F(p)\|\}.
\end{aligned}$$

By induction, we have

$$\|x_{n+1} - p\| \leq \{\|x_0 - p\|, \frac{\mu}{\tau} \mu \|F(p)\|\}.$$

Hence, $\{x_n\}$ is bounded and so are $\{T^{\lambda_n} x_n\}$, $\{F(x_n)\}$ and $\{S_n x_{n-1}\}$.

From (3.7), we have

$$
\begin{aligned}
&\|x_{n+1}-x_n\| \\
&= \Big\|\alpha_n T^{\lambda_n}x_n + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)S_i x_n - (\alpha_{n-1}T^{\lambda_{n-1}}x_{n-1} + \sum_{i=1}^{n-1}(\alpha_{i-1}-\alpha_i)S_i x_{n-1})\Big\| \\
&= \Big\|\alpha_n(T^{\lambda_n}x_n - T^{\lambda_n}x_{n-1}) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(S_i x_n - S_i x_{n-1}) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)S_i x_{n-1} \\
&\quad - \sum_{i=1}^{n-1}(\alpha_{i-1}-\alpha_i)S_i x_{n-1} + \alpha_n T^{\lambda_n}x_{n-1} - \alpha_{n-1}T^{\lambda_{n-1}}x_{n-1}\Big\| \\
&= \Big\|\alpha_n(T^{\lambda_n}x_n - T^{\lambda_n}x_{n-1}) + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(S_i x_n - S_i x_{n-1}) + (\alpha_{n-1}-\alpha_n)S_n x_{n-1} \\
&\quad + (\alpha_n-\alpha_{n-1})x_{n-1} + [\alpha_{n-1}(\lambda_{n-1}-\lambda_n) + (\alpha_{n-1}-\alpha_n)\lambda_n]\mu F(x_{n-1})\Big\| \\
&\le \alpha_n\|T^{\lambda_n}x_n - T^{\lambda_n}x_{n-1}\| + \sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|S_i x_n - S_i x_{n-1}\| + (\alpha_{n-1}-\alpha_n)\|S_n x_{n-1}\| \\
&\quad + (\alpha_{n-1}-\alpha_n)\|x_{n-1}\| + (|\lambda_{n-1}-\lambda_n| + (\alpha_{n-1}-\alpha_n))\mu\|F(x_{n-1})\| \\
&\le \alpha_n(1-\lambda_n\tau)\|x_n - x_{n-1}\| + (1-\alpha_n)\|x_n - x_{n-1}\| + (\alpha_{n-1}-\alpha_n)\|S_n x_{n-1}\| \\
&\quad + (\alpha_{n-1}-\alpha_n)\|x_{n-1}\| + (|\lambda_{n-1}-\lambda_n| + (\alpha_{n-1}-\alpha_n))\mu\|F(x_{n-1})\| \\
&\le (1-\alpha_n\lambda_n\tau)\|x_n - x_{n-1}\| + (\alpha_{n-1}-\alpha_n)M' + |\lambda_{n-1}-\lambda_n|M' \\
&\le (1-\alpha_n\lambda\tau)\|x_n - x_{n-1}\| + (\alpha_{n-1}-\alpha_n)M'' + |\lambda_{n-1}-\lambda_n|M', \qquad (3.8)
\end{aligned}
$$

where M' is a constant such that $M' \ge \sup\{\mu\|F(x_{n-1})\| + \|x_{n-1}\| + \|S_n x_{n-1}\|\}$.

Set $s_n = \|x_n - x_{n-1}\|$, $a_n = \alpha_n\lambda\tau$, $b_n = (\alpha_{n-1}-\alpha_n)M'$ and $c_n = |\lambda_n - \lambda_{n-1}|M'$. From (3.8) it follows that

$$s_{n+1} \le (1-a_n)s_n + b_n + c_n, \quad n \ge 1.$$

From the conditions (i), (ii) and (iii), we have

$$\sum_{n=1}^{\infty} a_n = \infty, \quad \lim_{n\to\infty}\frac{b_n}{a_n} = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} c_n < \infty.$$

From Lemma 2.2(2) we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \qquad (3.9)$$

Now, we prove that

$$\lim_{n\to\infty}\|x_n - S_i x_n\| = 0, \quad i \ge 1. \qquad (3.10)$$

It follows from (3.7) that

$$\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(x_n-S_ix_n)+x_{n+1}=(1-\alpha_n)x_n+\alpha_nT^{\lambda_n}x_n,$$

that is

$$\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)(x_n-S_ix_n)=\alpha_n(T^{\lambda_n}x_n-x_{n+1})+(1-\alpha_n)(x_n-x_{n+1}).$$

Hence, for any $p \in\in \bigcap_{i=1}^{\infty} Fix(S_i)$, we get

$$\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\langle x_n-S_ix_n, x_n-p\rangle=\alpha_n\langle T^{\lambda_n}x_n-x_{n+1}, x_n-p\rangle+(1-\alpha_n)\langle x_n-x_{n+1}, x_n-p\rangle. \tag{3.11}$$

Since each S_i is nonexpansive, by (2.1) we have

$$\|S_ix_n-x_n\|^2 \le 2\langle x_n-S_ix_n, x_n-p\rangle.$$

Hence, combining this inequality with (3.11), we get

$$\begin{aligned}\frac{1}{2}(\alpha_{i-1}-\alpha_i)\|S_ix_n-x_n\|^2 &\le \frac{1}{2}\sum_{i=1}^{n}(\alpha_{i-1}-\alpha_i)\|S_ix_n-x_n\|^2\\ &\le \alpha_n\langle T^{\lambda_n}x_n-x_{n+1}, x_n-p\rangle+(1-\alpha_n)\langle x_n-x_{n+1}, x_n-p\rangle,\end{aligned}$$

that is,

$$\|S_ix_n-x_n\|^2 \le \frac{2\alpha_n}{\alpha_{i-1}-\alpha_i}\langle T^{\lambda_n}x_n-x_{n+1}, x_n-p\rangle+\frac{1-\alpha_n}{\alpha_{i-1}-\alpha_i}\langle x_n-x_{n+1}, x_n-p\rangle.$$

Since $\{x_n\}$ and $\{T^{\lambda_n}x_n\}$ are both bounded, it follows from (3.9) and condition (i) that (3.10) holds.

By a similar process with the proof of Theorem 3.1, we can prove that

$$\limsup_{n\to\infty}\langle -Fx^*, x_n-x^*\rangle \le 0, \tag{3.12}$$

where x^* is the solution of $VI(F, C)$.

Finally, by Lemma 2.3 we have

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \Big\|[\alpha_n(T^{\lambda_n}x_n - T^{\lambda_n}x^*) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_n)(S_i x_n - x^*)] + \alpha_n(T^{\lambda_n}x^* - x^*)\Big\|^2 \\
&\le \Big\|\alpha_n(T^{\lambda_n}x_{n-1} - T^{\lambda_n}x^*) + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_n)(S_i x_n - x^*)\Big\|^2 + 2\alpha_n\langle T^{\lambda_n}x^* - x^*, x_{n+1} - x^*\rangle \\
&\le \alpha_n\|T^{\lambda_n}x_{n-1} - T^{\lambda_n}x^*\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_n)\|S_i x_n - x^*\|^2 + 2\alpha_n\langle T^{\lambda_n}x^* - x^*, x_n - x^*\rangle \\
&\le \alpha_n(1 - \lambda_n\tau)\|x_n - x^*\|^2 + \sum_{i=1}^{n}(\alpha_{i-1} - \alpha_i)\|x_n - x^*\|^2 + 2\alpha_n\langle T^{\lambda_n}x^* - x^*, x_n - x^*\rangle \\
&= \alpha_n(1 - \lambda_n\tau)\|x_n - x^*\|^2 + (1 - \alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\lambda_n\mu\langle -Fx^*, x_n - x^*\rangle \\
&= (1 - \alpha_n\lambda_n\tau)\|x_n - x^*\|^2 + 2\alpha_n\lambda_n\mu\langle -Fx^*, x_n - x^*\rangle \\
&\le (1 - \alpha_n\lambda\tau)\|x_n - x^*\|^2 + 2\alpha_n\lambda_n\mu\langle -Fx^*, x_n - x^*\rangle. \qquad (3.13)
\end{aligned}
$$

It follows from (3.12), (3.13), (i)-(iii) and Lemma 2.2(2) that we conclude that $\{x_n\}$ strongly converges to x^*, which completes the proof. □

The following two corollaries are obvious.

Corollary 3.5. *Let H be a real Hilbert space and $T : H \to H$ be a κ'-strict pseudo-contractions such that $C = Fix(T) \neq \emptyset$. Let $F : H \to H$ be a nonlinear mapping which is κ-Lipschitz continuous and η-strongly monotone on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$. Assume the sequences $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset [\lambda, 1)$ with $\lambda \in (0,1)$ satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\limsup_{n\to\infty}\alpha_n < 1$ *and* $\lim_{n\to\infty}(1 - \frac{\alpha_{n-1}}{\alpha_n}) = 1$*;*
(ii) $\sum_{n=1}^{\infty}|\lambda_n - \lambda_{n-1}| < \infty$.

Then the iterative sequence

$$x_0 \in H,\ x_{n+1} = \alpha_n(x_n - \lambda_n\mu F(x_n)) + (1 - \alpha_n)Sx_n, \quad n \ge 0,$$

where $Sx = \kappa' x + (1 - \kappa')Tx$ for all $x \in H$, converges strongly to the solution x^* of $VI(F, C)$, i.e., x^* solves the following variational inequality

$$\langle Fx^*, x - x^*\rangle \ge 0, \quad \forall x \in C.$$

Corollary 3.6. *Let H be a real Hilbert space and $\{T_i\} : H \to H$ be a countable family of nonexpansive mappings such that $C = Fix(T_i) \neq \emptyset$. Let $F : H \to H$ be a nonlinear mapping which is κ-Lipschitz continuous and η-strongly monotone on C. Take a fixed number $\mu \in (0, 2\eta/\kappa^2)$ and set $\alpha_0 = 1$. Assume the* *sequences* $\{\alpha_n\} \subset (0,1)$ *and* $\{\lambda_n\} \subset [\lambda, 1)$ *with* $\lambda \in (0,1)$ *satisfy the following conditions:*

(i) $\{\alpha_n\}$ *is strictly decreasing,* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\lim_{n\to\infty}(1 - \frac{\alpha_{n-1}}{\alpha_n}) = 1$*;*
(ii) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$.

Then the iterative sequence

$$x_0 \in H,\ x_{n+1} = \alpha_n(x_n - \lambda_n \mu F(x_n)) + \sum_{i=1}^{n} (\alpha_{i-1} - \alpha_i) T_i x_n, \quad n \geq 0,$$

converges strongly to the solution x^* *of* $VI(F, C)$*, i.e.,* x^* *solves the following variational inequality*

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

Remark 3.7. An simple example for the control sequences $\{\alpha_n\}$ and $\{\lambda_n\}$ in both Theorem 3.1 and Theorem 3.4 are that $\alpha_n = \frac{1}{n}$ and $\lambda_n = \lambda$, where $\lambda \in (0,1)$.

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References

[1] D. Kinderlehrer and G. Stampacchia, *An introduction to Variational inequalities and their applications*, Academic Press, New York, NY, 1980.

[2] I. Yamada, "The hybrid steepest descent method for the variational inequality problem over the intersection of fixed point sets of nonexpansive mappings," in: D. Butnarium Y. Censor, S. Reich (Eds.), Inherently Parallel Algorithm for Feasibility and Optimization, Elsevier, pp. 473–504, 2001.

[3] L.C. Zeng, Q.H. Ansari and S.Y. Wu, Strong convergence theorems of relaxed hybrid steepest-descent methods for variational inequalities, *Taiwanese Journal of Mathematics*, vol. 10, pp. 13–29, 2006.

[4] L.C. Zeng, N.C. Wong and J.C. Yao, "Convergence analysis of modified hybrid steepest-descent methods with variable parameters for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 132, pp. 51–69, 2007.

[5] L.C. Ceng, H.K. Xu and J.C. Yao, "A hybrid steepest-descent method for variational inequalities in Hilbert spaces," *Applicable Analysis*, vol. 87, pp. 575–589, 2008.

[6] H.K. Xu and T.H. Kim, "Convergence of hybrid steepest-descent methods for variational inequalities," *Journal of Optimization Theory and Applications*, vol. 119, pp. 185–201, 2003.

[7] S.H. Wang and G.D. Gu, "Hybrid steepest-descent methods with a countable family of nonexpansive mappings for variational inequalities in Hilbert spaces", *Applicalbe Analysis*, 2011, 1-12, iFirst, DOI: 10.1080./00036811.2011.559461.

[8] L.-C. Zeng and J.-C. Yao, "Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 64, pp. 2507–2515, 2006.

[9] P.E. Maingé, "Approximation methods for common fixed points of nonexpansive mappings in Hilbert spaces," *Journal Mathematics Analysis and Application* vol. 325, pp. 469–479, 2007.

[10] Y.J. Cho and S.H. Wang, "An Iterative Scheme with a Countable Family of Nonexpansive Mappings for Variational Inequality Problems in Hilbert Spaces,' *Journal of Inequalities and Applications*, vol. 2010, Article ID 687373, 14 pages.

[11] S.H. Wang, Y.J. Cho and X.L. Qin, "A New Iterative Method for Solving Equilibrium Problems and Fixed Point Problems for Infinite Family of Nonexpansive Mappings," *Fixed Point Theory and Applications*, vol. 2010, Article ID 165098, 18 pages.

[12] H.Y. Zhou, "Convergence theorems of fixed points for κ-strict pseudo-contractions in Hilbert spaces", *Nonlinear Analysis*, vol. 69, pp. 456–462, 2008.

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