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A Hilbert-Type Integral Inequality with the Integral in Whole Plane And Its Equivalent Forms

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Abstract

By establishing the weight function, we present a new Hilbert-type inequality with the homogeneous kernel of degree -3 and with the integral in whole plane, and also we put forward its equivalent forms.

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1 INTRODUCTION

If $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(x)}{x+y} dxdy < \pi \left\{ \int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx \right\}^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality, which has been extended by Hardy-Riesz as:

If $p > 1, \frac{1}{p} + \frac{1}{q} = 1$, $f(x)$, $g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)dx < \infty$ and $0 < \int_0^\infty g^q(x)dx < \infty$, then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} \left\{ \int_0^\infty f^p(x)dx \right\}^{1/p} \left\{ \int_0^\infty g^q(x)dx \right\}^{1/q}, \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ also is the best possible.

It attracted some attention in the recent years.Actually,inequalities (1.1) and(1.2) have many generalizations and variants. (1.1) has been strengthened by Yang and others. (including double series inequalities) [2-15].

In 2010 Zheng Zeng and Zitian Xie gave a new Hilbert-type integral inequality with the homogeneous kernel of degree 0 and with the integral in whole plane [2] :

In this paper,we present a new Hilbert-type inequality with the homogeneous kernel of degree -3 and with the integral in whole plane, and also we put forward its equivalent form.

In the following, we always suppose that $p > 1, 1/p + 1/q = 1; b > a > 0$.

2 SOME LEMMAS

Lemma 2.1 Define the weight functions as follow:

$$w(x) := \int_{-\infty}^{\infty} \frac{|x||y|dy}{\max\{|x|,|y|\}(y^2 + 2axy + b^2x^2)},$$
$$\widetilde{w}(y) := \int_{-\infty}^{\infty} \frac{y^2dx}{\max\{|x|,|y|\}(y^2 + 2axy + b^2x^2)}.$$

then

$$w(x) = \widetilde{w}(y) := k$$
$$= \frac{1}{2}\ln\frac{(1+b^2)^2 - 4a^4}{b^4} + \frac{\pi}{\sqrt{b^2-a^2}} + \frac{2a}{\sqrt{b^2-a^2}}\arctan\frac{a}{\sqrt{b^2-a^2}} +$$
$$+ \frac{a-1}{\sqrt{b^2-a^2}}\arctan\frac{1-a}{\sqrt{b^2-a^2}} - \frac{a+1}{\sqrt{b^2-a^2}}\arctan\frac{a+1}{\sqrt{b^2-a^2}}. \quad (2.1)$$

Proof We only prove that $w(x) = k$ for $x \in (-\infty, 0)$.

$$w(x) = \int_{-\infty}^{0} \frac{|x||y|dy}{\max\{|x|,|y|\}(y^2 + 2axy + b^2x^2)} +$$
$$+ \int_{0}^{\infty} \frac{|x||y|dy}{\max\{|x|,|y|\}(y^2 + 2axy + b^2x^2)} := w_1 + w_2,$$

setting $y = tx$,then

$$w_1 = \int_{-\infty}^{0} \frac{(-x)(-y)dy}{\max\{-x,-y\}(y^2 + 2axy + b^2x^2)} = \int_{0}^{\infty} \frac{tdt}{\max\{1,t\}(t^2 + 2at + b^2)}$$
$$= \int_{0}^{1} \frac{tdt}{t^2 + 2at + b^2} + \int_{1}^{\infty} \frac{dt}{t^2 + 2at + b^2}$$

$$= \frac{1}{2}\ln(t^2+2at+b^2)|_0^1 - \frac{a}{\sqrt{b^2-a^2}}\arctan\frac{t+a}{\sqrt{b^2-a^2}}|_0^1 + \\ + \frac{1}{\sqrt{b^2-a^2}}\arctan\frac{t+a}{\sqrt{b^2-a^2}}|_1^\infty$$

$$= \frac{1}{2}\ln\frac{1+2a+b^2}{b^2} + \frac{\pi}{2\sqrt{b^2-a^2}} + \frac{a}{\sqrt{b^2-a^2}}\arctan\frac{a}{\sqrt{b^2-a^2}} - \\ - \frac{a+1}{\sqrt{b^2-a^2}}\arctan\frac{a+1}{\sqrt{b^2-a^2}}$$

$$:= k_1$$

similarly, setting $y = -tx$, then

$$k_2 := w_2 \\ = \int_0^{+\infty}\frac{(-x)ydy}{\max\{-x,y\}(y^2+2axy+b^2x^2)} \\ = \int_0^{\infty}\frac{tdt}{\max\{1,t\}(t^2-2at+b^2)} \\ = \frac{1}{2}\ln\frac{1-2a+b^2}{b^2} + \frac{\pi}{2\sqrt{b^2-a^2}} + \frac{a}{\sqrt{b^2-a^2}}\arctan\frac{a}{\sqrt{b^2-a^2}} \\ + \frac{a-1}{\sqrt{b^2-a^2}}\arctan\frac{1-a}{\sqrt{b^2-a^2}}.$$

and

$$k = w(x) = w_1 + w_2 \\ = \frac{1}{2}\ln\frac{(1+b^2)^2-4a^4}{b^4} + \frac{\pi}{\sqrt{b^2-a^2}} + \frac{2a}{\sqrt{b^2-a^2}}\arctan\frac{a}{\sqrt{b^2-a^2}} + \\ + \frac{a-1}{\sqrt{b^2-a^2}}\arctan\frac{1-a}{\sqrt{b^2-a^2}} - \frac{a+1}{\sqrt{b^2-a^2}}\arctan\frac{a+1}{\sqrt{b^2-a^2}}.$$

Easily if $x \in (-\infty, 0)$, setting $x = -y/t$ and $x = y/t$, we have

$$\widetilde{w}(x) = \int_{-\infty}^{\infty}\frac{y^2dx}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)} \\ = \int_{-\infty}^{0}\frac{y^2dx}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)} + \int_0^{\infty}\frac{y^2dx}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)} \\ := \widetilde{w}_1(x) + \widetilde{w}_2(x).$$

and

$$\widetilde{w}_1(x) = \int_{-\infty}^{0} \frac{y^2 dx}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}$$
$$= \int_{\infty}^{0} \frac{y^2 d(y/t)}{\max\{-(y/t),-y\}(y^2+2a(y/t)y+b^2(y/t)^2)}$$
$$= \int_{0}^{\infty} \frac{tdt}{\max\{1,t\}(t^2+2at+b^2)} = k_1.$$

similarly,$\widetilde{w}_2(x) = k_2$, and $\widetilde{w}(x) = w(x) = k$.
the lemma is proved.

Lemma 2.2 For $\frac{q}{2} > \varepsilon > 0$, define both functions,$\widetilde{f}$ *and* $\widetilde{g}$, as follow:

$$\widetilde{f}(x) = \begin{cases} x^{-2\varepsilon/p}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } \ x \in [-1,1], \\ (-x)^{-2\varepsilon/p}, & \text{if } x \in (-\infty,-1); \end{cases}$$

$$\widetilde{g}(x) = \begin{cases} x^{1-2\varepsilon/q}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } \ x \in [-1,1], \\ (-x)^{1-2\varepsilon/q}, & \text{if } x \in (-\infty,-1), \end{cases}$$

then

$$I(\varepsilon) := \varepsilon \left\{\int_{-\infty}^{\infty} |x|^{-1}\widetilde{f}^p(x)dx\right\}^{1/p} \left\{\int_{-\infty}^{\infty} |x|^{-q-1}\widetilde{g}^q(x)dx\right\}^{1/q} = 1; \tag{2.2}$$

$$\widetilde{I}(\varepsilon) := \varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\widetilde{f}(x)\widetilde{g}(y)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dxdy$$
$$= k + o(1) \quad (\varepsilon \to 0^+). \tag{2.3}$$

Proof Easily,

$$I(\varepsilon) = 2\varepsilon \left\{\int_{1}^{\infty} x^{-1}x^{-2\varepsilon}dx\right\}^{1/p} \left\{\int_{1}^{\infty} x^{-1}x^{-2\varepsilon}dx\right\}^{1/q} = 1;$$

Let $y = -Y$,using $\widetilde{f}(-x) = \widetilde{f}(x)$, $\widetilde{g}(-x) = \widetilde{g}(x)$, and

$$\widetilde{f}(-x) \int_{-\infty}^{\infty} \frac{\widetilde{g}(y)dy}{\max\{|x|,|y|\}(y^2-2axy+b^2x^2)}$$

$$= \widetilde{f}(x) \int_{-\infty}^{\infty} \frac{\widetilde{g}(Y)dY}{\max\{|x|,|Y|\}(Y^2+2axY+b^2x^2)}$$

we have that $\widetilde{f}(x)\int_{-\infty}^{\infty}\frac{\widetilde{g}(y)dy}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}$ is an even function ,then

$$\begin{aligned}
\widetilde{I}(\varepsilon) &= 2\varepsilon\int_0^{\infty}\widetilde{f}(x)\left(\int_{-\infty}^{\infty}\frac{\widetilde{g}(y)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dy\right)dx\\
&= 2\varepsilon\left[\int_1^{\infty}x^{-\frac{2\varepsilon}{p}}\left(\int_{-\infty}^{-1}\frac{(-y)^{1-\frac{2\varepsilon}{q}}}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dy\right)dx\right.\\
&\quad\left.+\int_1^{\infty}x^{-\frac{2\varepsilon}{p}}\left(\int_1^{\infty}\frac{y^{1-\frac{2\varepsilon}{q}}}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dy\right)dx\right]\\
&:= I_1+I_2.
\end{aligned}$$

Setting $y=tx$ then

$$\begin{aligned}
I_1 &= 2\varepsilon\left[\int_1^{\infty}x^{-\frac{2\varepsilon}{p}}\left(\int_1^{\infty}\frac{y^{1-\frac{2\varepsilon}{q}}}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dy\right)dx\right]\\
&= 2\varepsilon\left[\int_1^{\infty}x^{-1-2\varepsilon}\left(\int_{\frac{1}{x}}^{\infty}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt\right)dx\right]\\
&= 2\varepsilon\left[\int_1^{\infty}x^{-1-2\varepsilon}\left(\int_1^{\infty}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt\right)dx\right.\\
&\quad\left.+\int_1^{\infty}x^{-1-2\varepsilon}\left(\int_{\frac{1}{x}}^{1}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt\right)dx\right]\\
&= \int_1^{\infty}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt\\
&\quad+2\varepsilon\int_0^{1}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}\left(\int_{\frac{1}{t}}^{\infty}x^{-1-2\varepsilon}dx\right)dt\\
&= \int_1^{\infty}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt+\int_0^{1}\frac{t^{1+\frac{2\varepsilon}{p}}}{\max\{1,t\}(t^2+2at+b^2)}dt\\
&= \int_0^{\infty}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt+\int_0^{1}(t^{\frac{2\varepsilon}{p}}-t^{-\frac{2\varepsilon}{q}})\frac{t}{\max\{1,t\}(t^2+2at+b^2)}dt\\
&= \int_0^{\infty}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt+\eta(\varepsilon)(\text{ there }\lim_{\varepsilon\to0^+}\eta(\varepsilon)=0)\\
&= k_1+o(1)\ \ (\varepsilon\to0^+).
\end{aligned}$$

The reason for the last equation is that,$\exists M\in\mathbf{R}_+$,such that

$$\int_0^{\infty}\frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt$$

$$< \int_0^1 \frac{1}{\max\{1,t\}(t^2+2at+b^2)}dt + \int_1^\infty \frac{t}{\max\{1,t\}(t^2+2at+b^2)}dt < M$$

and by using the theorem of control convergence, we have the conclusion as follows:

$$\lim_{\varepsilon\to 0^+}\int_0^\infty \frac{t^{1-\frac{2\varepsilon}{q}}}{\max\{1,t\}(t^2+2at+b^2)}dt = \int_0^\infty \frac{t}{\max\{1,t\}(t^2+2at+b^2)}dt = k_1.$$

and we have$I_1 \to k_1 \quad (\varepsilon \to 0^+)$.

Similarly $I_2 \to k_2 \quad (\varepsilon \to 0^+)$.The lemma is proved.

Lemma 2.3 We have

$$J := \int_{-\infty}^\infty |y|^{2p-1}\left(\int_{-\infty}^\infty \frac{f(x)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dx\right)^p dy$$
$$\le k^p \int_{-\infty}^\infty |x|^{-1} f^p(x)dx. \tag{2.4}$$

Proof By lemma 2.2,we find

$$\begin{aligned}
&\left(\int_{-\infty}^\infty \frac{f(x)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dx\right)^p \\
&= \left[\int_{-\infty}^\infty \frac{1}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}|y|^{1/p}f(x)|y|^{-1/p}dx\right]^p \\
&\le \int_{-\infty}^\infty \frac{|y|}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}f^p(x)dx \\
&\quad \times \left(\int_{-\infty}^\infty \frac{|y|^{1-q}}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dx\right)^{p-1} \\
&= k^{p-1}|y|^{-2p+1}\int_{-\infty}^\infty \frac{|y|}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}f^p(x)dx,
\end{aligned}$$

$$\begin{aligned}
J &\le k^{p-1}\int_{-\infty}^\infty\left[\int_{-\infty}^\infty \frac{|y|}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}f^p(x)dx\right]dy \\
&= k^{p-1}\int_{-\infty}^\infty\left[\int_{-\infty}^\infty \frac{|y|}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dy\right]f^p(x)dx \\
&= k^p \int_{-\infty}^\infty |x|^{-1}f^p(x)dx.
\end{aligned}$$

3 MAIN RESULTS

Theorem 3.1 If both functions, $f(x)$ and $g(x)$,are nonnegative measurable functions, and satisfy

$0 < \int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx < \infty$ and $0 < \int_{-\infty}^{\infty} |x|^{-q-1} g^q(x)dx < \infty$. then,

$$I^* := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)} dxdy$$
$$< k \left(\int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx\right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{-q-1} g^q(x)dx\right)^{1/q}, \tag{3.1}$$

and

$$J = \int_{-\infty}^{\infty} |y|^{2p-1} (\int_{-\infty}^{\infty} \frac{f(x)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)} dx)^p dy$$
$$< k^p \int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx. \tag{3.2}$$

Inequalities (3.1)and (3.2) are equivalent,and where the constant factors k and k^p are the best possible.

Proof If (2.4) takes the form of equality for some $y \in (-\infty,0)\cup(0,\infty)$, then there exists constants M and N, such that they are not all zero, and

$$M|y|f^p(x) = N|y|^{-q+1} \text{ a.e. in } (-\infty,\infty).$$

Hence, there exists a constant C, such that

$$Mf^p(x) = N|y|^{-q} = C \qquad \text{a.e. in} \quad (-\infty,\infty).$$

We claim that $M = 0$. In fact, if $M \neq 0$, then $|x|^{-1}f^p(x) = \frac{C}{M|x|}$ a.e. in $(-\infty,\infty)$ which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx < \infty$. In the same way, we claim that $N = 0$. This is too a contradiction and hence by (2.4), we have (3.2).

By Hölder's inequality with weight and (3.2), we have,

$$I^* = \int_{-\infty}^{\infty} \left[|y|^{1+\frac{1}{q}} \int_{-\infty}^{\infty} \frac{f(x)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)} dx\right] \left[|y|^{-1-\frac{1}{q}} g(y)\right] dy$$
$$\leq J^{\frac{1}{p}} \left(\int_{-\infty}^{\infty} |y|^{-q-1} g^q(y)dy\right)^{1/q}. \tag{3.3}$$

Using (3.2),we have (3.1).

Setting

$$g(y) = |y|^{2p-1} (\int_{-\infty}^{\infty} \frac{f(x)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)} dx)^{p-1},$$

then $J = \int_{-\infty}^{\infty} |y|^{-q-1} g^q(y)dy$ by (2.4)we have $J < \infty$ .if $J = 0$ then (3.2)is proved;if $0 < J < \infty$, by (3.1),we obtain

$$0 < \int_{-\infty}^{\infty} |y|^{-q-1} g^q(y)dy = J = I^*$$

$$< k\left(\int_{-\infty}^{\infty}|x|^{-1}f^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^{q}(x)dx\right)^{1/q},$$

$$\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^{q}(x)dx\right)^{1/p}=J^{1/p}< k\left(\int_{-\infty}^{\infty}|x|^{-1}f^{p}(x)dx\right)^{1/p}.$$

Inequalities (3.1)and (3.2) are equivalent.

If the constant factor k in (3.1) is not the best possible, then there exists a positive h (with $h < k$),such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{\max\{|x|,|y|\}(y^2+2axy+b^2x^2)}dxdy$$

$$< h\left(\int_{-\infty}^{\infty}|x|^{-1}f^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^{q}(x)dx\right)^{1/q}. \tag{3.4}$$

For $\varepsilon > 0$,by (3.4),using lemma 2.2 ,we have

$$k+o(1)<\varepsilon h\left(\int_{-\infty}^{\infty}|x|^{-1}\widetilde{f}^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}\widetilde{g}^{q}(x)dx\right)^{1/q}=h.$$

Hence we find, $k + o(1) < h$. For $\varepsilon \to 0^+$, it follows that $k \leq h$, which contradicts the fact that $h < k$. Hence the constant k in (3.1) is the best possible. Thus we complete the prove of the theorem.

Remarks 1)For $a = \cos\theta$, $\theta \in (0, \pi)$, $b = 1$,then

$$k(\theta)=\frac{1}{2}\ln[4(1+\cos^2\theta)\sin^2\theta]+\frac{\pi}{\sin\theta}+2(\frac{\pi}{2}-\theta)\cot\theta-\frac{\theta}{2}\tan\frac{\theta}{2}-\frac{\pi-\theta}{2}\cot\frac{\theta}{2}$$
$$=\ln(2\sin\theta)+\frac{\pi}{\sin\theta}+(\pi-2\theta)\cot\theta-\frac{\pi}{2}\cot\frac{\theta}{2}-\frac{\theta}{2}\tan\frac{\theta}{2}+\frac{\theta}{2}\cot\frac{\theta}{2}$$
$$=\ln(2\sin\theta)+\frac{\pi}{\sin\theta}+(\pi-\theta)\cot\theta-\frac{\pi}{2}\cot\frac{\theta}{2}$$

and we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{\max\{|x|,|y|\}(y^2+2xy\cos\theta+x^2)}dxdy$$

$$< k(\theta)\left(\int_{-\infty}^{\infty}|x|^{-1}f^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^{q}(x)dx\right)^{1/q}, \tag{3.7}$$

2)If $\theta = \frac{\pi}{3}$, in (3.7),then $k(\frac{\pi}{3}) = \frac{1}{2}\ln 3 + \frac{7\pi}{6\sqrt{3}}$ we have the following particular result:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{\max\{|x|,|y|\}(y^2+xy+x^2)}dxdy$$

$$< (\frac{1}{2}\ln\frac{15}{4}+\frac{7\pi}{6\sqrt{3}})\left(\int_{-\infty}^{\infty}|x|^{-1}f^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^{q}(x)dx\right)^{1/q}, \tag{3.8}$$

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