Mathematica Aeterna, Vol. 1, 2011, no. 08, 587 - 598

The existence of NBIBDs with $k_1 = 6$ and $\lambda_1 = 5$ [1]

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Abstract

In this paper we focus on the existence of nested balanced incomplete block designs with $k_1 = 6$ and $\lambda_1 = 5$. The necessary conditions for the existences of a $(v, 6, 2, 5, 1)$-NBIBD and a $(v, 6, 3, 5, 2)$-NBIBD are the same, i.e., $v \equiv 0, 1 \pmod 3$. For $(v, 6, 2, 5, 1)$-NBIBD, the necessary conditions are also sufficient. For $(v, 6, 3, 5, 2)$-NBIBD, the necessary conditions are sufficient except for $v \in \{6, 10\}$, and possibly for $v \in \{22, 34, 39, 45, 48, 51, 54, 60, 64, 69, 70, 72, 75, 82, 87, 88, 90, 102, 111, 148\}$.

Keywords: nested balanced incomplete block design; perfect Mendelsohn design; generalized whist tournament design
Mathematics Subject Classification: 05B07

1 Introduction

A (v, k, λ)-*balanced incomplete block design* (briefly (v, k, λ)-BIBD) is a pair $(V, \mathcal{B})$, where V is a set of v elements, $\mathcal{B}$ is a collection of k-subsets, called *blocks*, of V such that every pair of distinct elements of V occurs in exactly λ blocks of $\mathcal{B}$.

Let k_1 be a positive integer. Let k_2 be a submultiple of k_1 and $k_2 > 1$. If each block of a (v, k_1, λ_1)-BIBD $(V, \mathcal{B}_1)$ can be partitioned into k_1/k_2 *sub-blocks*

[1]This work was supported by the Fundamental Research Funds for the Central Universities (T. Feng and Y. Chang), the NSFC under Grant 10901016 (T. Feng), and the NSFC under Grant 61071221 (Y. Chang).

of size k_2, and the collection of all sub-blocks constitutes a (v, k_2, λ_2)-BIBD $(V, \mathcal{B}_2)$, then the triple $(V, \mathcal{B}_1, \mathcal{B}_2)$ is said to be a *nested balanced incomplete block design*, denoted by a $(v, k_1, k_2, \lambda_1, \lambda_2)$-NBIBD.

The blocks of a NBIBD will be displayed in the form of $\{x_1, \ldots, x_{k_2}; y_1, \ldots, y_{k_2}; \ldots\}$ with the semicolons separating the sub-blocks. It is easy to verify that the necessary conditions for the existence of a $(v, k_1, k_2, \lambda_1, \lambda_2)$-NBIBD are $k_2 \mid k_1$, $\lambda_1(v-1) \equiv 0 \pmod{(k_1-1)}$, $\lambda_2(v-1) \equiv 0 \pmod{(k_2-1)}$, $\lambda_1 v(v-1) \equiv 0 \pmod{k_1(k_1-1)}$, $\lambda_2 v(v-1) \equiv 0 \pmod{k_2(k_2-1)}$, $\lambda_2(k_1-1) = \lambda_1(k_2-1)$. NBIBDs were first introduced in 1967 by Preece [10]. Its study has been motivated by a biological experimental background on the effect of inoculating plants with virus [6, 7]. For more information on NBIBDs, the interested reader may refer to [8, 9] and the references therein.

NBIBDs are closely related to generalized whist tournament designs. A generalized whist tournament design is a resolvable BIBD. A BIBD is *resolvable* if its blocks can be partitioned into parallel classes; a parallel class is a set of point-disjoint blocks whose union is the set of all points. Let t, k be positive integers and k be divided by t. Let $v \equiv 0, 1 \pmod{k}$. A (t, k) *generalized whist tournament design* on v players, denoted by (t, k)GWhD(v), is a (near) resolvable $(v, k, k-1)$-BIBD on the v-set X whose blocks can be partitioned into k/t *sub-blocks* in such a way that (1) each sub-block consists of t players; (2) players appearing in the same sub-block are designated as partners; (3) players appearing in the same block but not in the same sub-block are designated as opponents; (4) each player partners every other player exactly $t-1$ times and each player opposes every other player exactly $k-t$ times.

Lemma 1.1 ([5]) *A (t, k)GWhD(v) is a $(v, k, t, k-1, t-1)$-NBIBD with the additional property of resolvability or near-resolvability.*

NBIBDs are also closely related to perfect Mendelsohn designs. A set of k distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be *cyclically ordered* by $a_1 < a_2 < \cdots < a_k < a_1$ and the elements a_i, a_{i+t} are said to be *t-apart* in a cyclic k-tuple $(a_1, a_2, \ldots, a_k)$, where $i+t$ is taken modulo k. A (v, k, λ)-*holey perfect Mendelsohn design* (briefly (v, k, λ)-HPMD) is a triple $(X, \mathcal{H}, \mathcal{B})$ which satisfies (1) $\mathcal{H}$ is a partition of v-set X into subsets called *holes*; (2) $\mathcal{B}$ is a family of cyclically ordered k-subsets of X (called *blocks*) such that a hole and a block contain at most one common block; (3) every ordered pair of elements from distinct holes are t-apart in exactly λ blocks for each $t = 1, 2, \ldots, k-1$. If $\mathcal{H}$ contains u_i groups of size g_i for $1 \le i \le r$, then we call $g_1^{u_1} g_2^{u_2} \cdots g_r^{u_r}$ the *group type* (or *type*) of the HPMD. A (v, k, λ)-HPMD of type $1^{v-n} n^1$ is called an *incomplete perfect Mendelsohn design*, denoted by a (v, n, k, λ)-IPMD. A (v, k, λ)-HPMD of type 1^v is simply written as a (v, k, λ)-PMD.

Lemma 1.2 *Suppose that there exists a $(v, k, 1)$-PMD, then there exists a* $(v, k, t, k-1, t-1)$-NBIBD *for each t satisfying $t \mid k$.*

Proof Assume that there exists a $(v, k, 1)$-PMD. Let $k = et$ and $(a_1, \ldots, a_k)$ be a block of the $(v, k, 1)$-PMD. Rearrange this block as follows:

$$\{a_1, a_{1+e}, \ldots, a_{1+(t-1)e}; a_2, a_{2+e}, \ldots, a_{2+(t-1)e}; \ldots; a_e, a_{2e}, \ldots, a_k\}.$$

Apply the above procedure to each block of the PMD. Then we have a $(v, k, t, k-1, t-1)$-NBIBD. □

We quote the following results for later use.

Theorem 1.3 ([1, Theorem 10.1]) *The necessary conditions for the existence of a $(v, 6, 1)$-PMD, namely, $v \equiv 0, 1 \pmod 3$ and $v \geq 6$, are sufficient except for $v = 6$, and possibly for*

(1) $v \in \{12, 18, 24, 30, 48, 54, 60, 72, 84, 90, 96, 102, 108, 114, 132, 138, 150, 162, 168, 180, 192, 198\}$;

(2) $v \equiv 3 \pmod 6$ and $v \in [9, 135] \cup [153, 183] \cup \{207, 213, 219, 237, 243, 255, 297, 375, 411, 435,$
$453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}$;

(3) $v \equiv 4 \pmod 6$ and $v \in \{10, 16, 22, 34\} \cup [52, 148]$.

Theorem 1.4 ([1, Theorem 3.4]) *Suppose that $q = 4t + 1$ is a prime power.*

(1) *If t is odd and $t \neq 1$, then there exists a $(q + t, t, 6, 1)$-IPMD.*

(2) *If $t \equiv 2 \pmod 4$ and $t \neq 2$, then there exists a $(q + t, t, 6, 1)$-IPMD.*

(3) *If $t \equiv 4 \pmod 8$, then there exists a $(q + t, t, 6, 1)$-IPMD.*

In this paper, we shall focus our attention on the problem of the existence of NBIBDs with $k_1 = 6, \lambda_1 = 5$. It is easy to verify that the necessary condition for the existence of a $(v, 6, 2, 5, 1)$- NBIBD and a $(v, 6, 3, 5, 2)$-NBIBD is $v \equiv 0, 1 \pmod 3$. As the main results, we are to prove the following theorems.

Theorem 1.5 *The necessary condition for the existence of a $(v, 6, 2, 5, 1)$-NBIBD, namely, $v \equiv 0, 1 \pmod 3$ and $v \geq 6$, is also sufficient.*

Theorem 1.6 *The necessary condition for the existence of a $(v, 6, 3, 5, 2)$-NBIBD, namely, $v \equiv 0, 1 \pmod 3$ and $v \geq 6$, is also sufficient, except for $v \in \{6, 10\}$, and possibly for $v \in \{22, 34, 39, 45, 48, 51, 54, 60, 64, 69, 70, 72, 75, 82, 87, 88, 90, 102, 111, 148\}$.*

2 Basic design constructions

In this section we shall give some recursive constructions for NBIBDs. First we need to introduce some auxiliary designs.

Let X be a set of v elements. A $(v, k_1, k_2, \lambda_1, \lambda_2)$-*nested group divisible design* (briefly, $(v, k_1, k_2, \lambda_1, \lambda_2)$-NGDD) is a quadruple $(X, \mathcal{H}, \mathcal{B}_1, \mathcal{B}_2)$ satisfying the following properties: (1) $\mathcal{H}$ is a partition of X into subsets (called *groups or holes*); (2) $\mathcal{B}_1$ is a collection of subsets of X (called *blocks*), each of size k_1; (3) each of blocks of $\mathcal{B}_1$ can be partitioned into k_1/k_2 *sub-blocks* of size k_2, and denote the collection of all sub-blocks of $\mathcal{B}_1$ by $\mathcal{B}_2$; (4) every 2-subset of X is either contained in exactly λ_1 block of $\mathcal{B}_1$ or in exactly one group, but not in both; (5) every 2-subset of X is either contained in exactly λ_2 block of $\mathcal{B}_2$ or in exactly one group, but not in both. If $\mathcal{H}$ contains u_i groups of size g_i for $1 \le i \le r$, then we call $g_1^{u_1} g_2^{u_2} \cdots g_r^{u_r}$ the *group type* (or *type*) of the NGDD. Especially a $(v, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $1^{v-n}n^1$ is called an *incomplete nested balanced incomplete block design*, denoted by $(v, n, k_1, k_2, \lambda_1, \lambda_2)$-INBIBD.

The following construction is simple but very useful.

Construction 2.1 (Filling construction) *Suppose that there is a* $(v, k_1, k_2, \lambda_1, \lambda_2)$*-NGDD of type* $\{m_1, m_2, \ldots, m_s\}$*, where* $v = \sum_{i=1}^{s} m_i$*. If there exist an* $(m_i + w, w, k_1, k_2, \lambda_1, \lambda_2)$*-INBIBD for each* $1 \le i \le s-1$*, and an* $(m_s + w, k_1, k_2, \lambda_1, \lambda_2)$*-NBIBD, then there is a* $(v + w, k_1, k_2, \lambda_1, \lambda_2)$*-NBIBD.*

Let K be a set of positive integers. A *group divisible design* (GDD) K-GDD is a triple $(X, \mathcal{G}, \mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of a finite set X into subsets (called *groups*); (2) $\mathcal{A}$ is a set of subsets of X (called *blocks*), each of cardinality from K, such that every 2-subset of X is either contained in exactly one block or in exactly one group, but not in both. If $\mathcal{G}$ contains u_i groups of size g_i for $1 \le i \le r$, then we call $g_1^{u_1} g_2^{u_2} \cdots g_r^{u_r}$ the *group type* (or *type*) of the GDD. If $K = \{k\}$, we write $\{k\}$-GDD as k-GDD.

A *transversal design* (TD) TD(k, n) is a GDD of group type n^k and block size k. It is well known that a TD(k, n) is equivalent to $k-2$ mutually orthogonal Latin squares (MOLS) of order n. The following lemma will be used later.

Lemma 2.2 [2]

(1) *A TD*$(6, m)$ *exists for all integers* $m > 4$ *except for* $m = 6$ *and possibly for* $m \in \{10, 14, 18, 22\}$*.*

(2) *A TD*$(q + 1, q)$ *exists for any prime power* q*.*

In recursive constructions of GDDs and PBDs, the weighting technique and Wilson's Fundamental Construction [11] are frequently used. Similar techniques are also available for constructing NGDDs.

Construction 2.3 (Weighting construction) *Suppose that $(X, \mathcal{G}, \mathcal{B})$ is a K-GDD and w is a function from X to $Z^+ \cup \{0\}$. Suppose that there exists a $(\sum_{x\in B} w(x), k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $\{w(x)|x \in B\}$ for every $B \in \mathcal{B}$. Then there exists a $(v, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $\{\sum_{x\in G} w(x)|G \in \mathcal{G}\}$, where $v = \sum_{x\in X} w(x)$.*

Proof For every $x \in X$, let $S(x)$ be a set of $\omega(x)$ "copies" of x. For every block $B \in \mathcal{B}$, construct a $(\sum_{x\in B} w(x), k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $\{w(x)|x \in B\}$ $(\bigcup_{x\in B} S_x, \{S_x|x \in B\}, \mathcal{A}_\mathcal{B}, \mathcal{C}_\mathcal{B})$. Denote

$$Y = \bigcup_{x\in X} S_x, \qquad \mathcal{H} = \{\bigcup_{x\in G} S_x|G \in \mathcal{G}\} \qquad \mathcal{A} = \bigcup_{B\in\mathcal{B}} \mathcal{A}_B, \qquad \mathcal{C} = \bigcup_{B\in\mathcal{B}} \mathcal{C}_B.$$

Then it is readily checked that $(Y, \mathcal{H}, \mathcal{A}, \mathcal{C})$ is the required NGDD. □

Construction 2.4 (Inflating construction) *If there exist a $(\sum_{i=1}^{s} g_i h_i, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $g_1^{h_1} g_2^{h_2} \cdots g_s^{h_s}$ and a TD(k_1, m), then an $(m\sum_{i=1}^{s} g_i h_i, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $(g_1m)^{h_1}(g_2m)^{h_2} \cdots (g_sm)^{h_s}$ exists.*

Proof Let $(X, \mathcal{G}, \mathcal{B}_1, \mathcal{B}_2)$ be a $(\sum_{i=1}^{s} g_i h_i, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $g_1^{h_1} g_2^{h_2} \cdots g_s^{h_s}$. Denote $M = \{1, 2, \ldots, m\}$. Let $X^* = X \times M$. For each $B \in \mathcal{B}_1$, construct a TD(k_1, m) on $B \times M$, whose block set is denoted by $\mathcal{A}_B$. Let $B = \bigcup_{j=1}^{k_1/k_2} B_j$ and $B_j \in \mathcal{B}_2$. Then for each $1 \le j \le k_1/k_2$, we have a TD(k_2, m) on $B_j \times M$, which is obtained by removing elements of $(B \setminus B_j) \times M$ in the TD(k_1, m). Denote its block set by $\mathcal{C}_B^{(j)}$. Let $\mathcal{C}_B = \bigcup_{j=1}^{k_1/k_2} \mathcal{C}_B^{(j)}$. Let

$$\mathcal{H} = \{\bigcup_{x\in G}(\{x\} \times M)|G \in \mathcal{G}\}, \qquad \mathcal{A} = \bigcup_{B\in\mathcal{B}_1} \mathcal{A}_B, \qquad \mathcal{C} = \bigcup_{B\in\mathcal{B}_1} \mathcal{C}_B.$$

It is readily checked that $(X^*, \mathcal{H}, \mathcal{A}, \mathcal{C})$ is the required NGDD. □

Similar arguments to that in Lemma 1.2, we have

Lemma 2.5 *If there exists a $(v, k, 1)$-HPMD of type $g_1^{h_1} g_2^{h_2} \cdots g_s^{h_s}$, then there exists a $(v, k, t, k-1, t-1)$-NGDD of type $g_1^{h_1} g_2^{h_2} \cdots g_s^{h_s}$ for each t satisfying $t \mid k$.*

3 Existence of $(v, 6, 2, 5, 1)$-NBIBDs

Lemma 3.1 [9] *There exists a $(v, 6, 2, 5, 1)$-NBIBD for $v \in \{6, 7, 9, 10, 12, 13, 15, 16\}$.*

By Lemma 1.1, a (t, k)GWhD(v) implies a $(v, k, t, k-1, t-1)$-NBIBD. Since for any positive integer $v \equiv 0 \pmod 6$ and $v \notin \{18, 108, 132, 174, 264\}$, a $(2, 6)$GWhD(v) exists [3], we have

Lemma 3.2 *There exists a* $(v, 6, 2, 5, 1)$*-NBIBD for any positive integer* $v \equiv 0 \pmod 6$ *and* $v \notin \{18, 108, 132, 174, 264\}$.

By Lemma 1.2, if there exists a $(v, k, 1)$-PMD, then there exists a $(v, k, t, k-1, t-1)$-NBIBD for each t satisfying $t \mid k$. Combining the results of Theorem 1.3, Lemmas 3.1 and 3.2, we have

Lemma 3.3 *There is a* $(v, 6, 2, 5, 1)$*-NBIBD for any integer* $v \equiv 0, 1 \pmod 3$ *and* $v \geq 6$ *except for*

(1) $v \in \{18, 108, 132\}$;

(2) $v \equiv 3 \pmod 6$ and $v \in [21, 135] \cup [153, 183] \cup \{207, 213, 219, 237, 243, 255, 297, 375, 411, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}$;

(3) $v \equiv 4 \pmod 6$ and $v \in \{22, 34\} \cup [52, 148]$.

Lemma 3.4 *There exists a* $(v, 6, 2, 5, 1)$*-NBIBD for* $v \in \{18, 21, 22, 27, 33, 34, 39\}$.

Proof For $v \in \{21, 22, 33, 39\}$, let $X = Z_v$ and develop the following base blocks:

$v = 21$:	$\{0, 2; 6, 15; 12, 16\}$,	$\{0, 7; 11, 19; 3, 6\}$,	$\{0, 15; 16, 17; 7, 18\}$,
	$\{0, 16; 2, 7; 9, 14\}^*$,		$(+1 \bmod 21)$;
$v = 22$:	$\{0, 5; 1, 2; 10, 13\}$,	$\{0, 2; 3, 18; 16, 6\}$,	$\{0, 8; 6, 15; 1, 5\}$,
	$\{0, 6; 11, 17; 2, 13\}^*$,		$(+1 \bmod 22)$;
$v = 33$:	$\{0, 3; 12, 18; 17, 28\}$,	$\{0, 28; 2, 16; 3, 29\}$,	$\{0, 16; 2, 3; 27, 12\}$,
	$\{0, 20; 26, 16; 28, 30\}$,	$\{0, 21; 20, 28; 6, 30\}$,	$\{0, 4; 11, 15; 22, 26\}^*$,
			$(+1 \bmod 33)$;
$v = 39$:	$\{0, 4; 3, 17; 19, 32\}$,	$\{0, 1; 30, 12; 5, 36\}$,	$\{0, 9; 8, 25; 3, 27\}$,
	$\{0, 2; 3, 31; 7, 23\}$,	$\{0, 3; 9, 28; 30, 1\}$,	$\{0, 5; 9, 16; 22, 34\}$,
	$\{0, 6; 13, 19; 26, 32\}^*$,		$(+1 \bmod 39)$.

For $v \in \{18, 27, 34\}$, let $X = Z_{v-1} \cup \{\infty\}$ and develop the following base blocks:

$v = 18$:	$\{\infty, 12; 0, 9; 2, 8\}$,	$\{0,15;1,2;4,7\}$,	$\{0, 12; 1, 8; 3, 7\}$,
			$(+1 \bmod 17)$;
$v = 27$:	$\{\infty, 0; 7, 13; 11, 21\}$,	$\{0, 3; 6, 14; 24, 17\}$,	$\{0, 9; 20, 25; 16, 18\}$,
	$\{0, 22; 1, 2; 8, 23\}$,	$\{0, 13; 16, 4; 3, 17\}^*$,	$(+1 \bmod 26)$;
$v = 34$:	$\{\infty, 0; 30, 9; 7, 29\}$,	$\{0, 16; 32, 3; 24, 18\}$,	$\{0, 30; 9, 28; 15, 23\}$,
	$\{0, 31; 3, 4; 12, 32\}$,	$\{0, 7; 23, 13; 21, 30\}$,	$\{0, 15; 11, 26; 22, 4\}^*$,
	$\{0, 5; 11, 16; 22, 27\}^*$,		$(+1 \bmod 33)$,

where $\infty + 1 = \infty$. When $v \in \{21, 33, 39\}$, the base blocks with a * will generate $v/3$ distinct blocks under Z_v; when $v = 22$, the base block with a * will generate 11 distinct blocks under Z_{22}; when $v = 27$, the base block with a * will generate 13 distinct blocks under Z_{26}; when $v = 34$, the base blocks with a * will generate 11 distinct blocks under Z_{33}. □

Lemma 3.5 *There exists a* $(11, 2, 6, 2, 5, 1)$*-INBIBD.*

Proof Let $X = Z_9 \cup \{\infty_1, \infty_2\}$ with $\{\infty_1, \infty_2\}$ as the hole. Develop the two base blocks: $\{0, 1; 4, 8; 3, \infty_1\}$ and $\{0, 2; 5, 8; 7, \infty_2\}$ by $+1$ modulo 9 to obtain all the blocks, where $\infty_i + 1 = \infty_i$ for each $i = 1, 2$. □

Lemma 3.6 *There exists a* $(69, 12, 6, 2, 5, 1)$*-INBIBD.*

Proof Let $X = Z_{57} \cup \{\infty_1, \infty_2, \ldots, \infty_{12}\}$ with $\{\infty_1, \infty_2, \ldots, \infty_{12}\}$ as the hole. Develop the following base blocks by $+1$ modulo 57 to obtain all the blocks, where $\infty_i + 1 = \infty_i$ for each $1 \le i \le 12$.

$\{19, 1; 8, 38; 37, 11\}$, $\{0, 55; 33, 24; 40, \infty_1\}$, $\{6, 28; 10, 51; 0, \infty_2\}$,
$\{0, 42; 30, 13; 56, \infty_3\}$, $\{15, 19; 51, 18; 38, \infty_4\}$, $\{0, 32; 42, 30; 37, \infty_5\}$,
$\{27, 33; 19, 38; 6, \infty_6\}$, $\{0, 21; 9, 32; 49, \infty_7\}$, $\{0, 10; 9, 14; 3, \infty_8\}$,
$\{23, 24; 0, 7; 39, \infty_9\}$, $\{0, 11; 21, 13; 9, \infty_{10}\}$, $\{0, 29; 9, 52; 3, \infty_{11}\}$,
$\{39, 26; 24, 4; 0, \infty_{12}\}$, $\{0, 3; 19, 22; 38, 41\}^*$.

The base block with a * will generate 19 distinct blocks under Z_{57}. □

Lemma 3.7 *There exists a* $(v, 6, 2, 5, 1)$*-NBIBD for* $v \in \{45, 52, 58, 69\}$*.*

Proof A $(45, 6, 6, 1)$-IPMD, a $(52, 9, 6, 1)$-IPMD and a 6-HPMD of type $7^7 9^1$ are constructed explicitly in [1]. Then by Lemma 2.5, there is a $(45, 6, 6, 2, 5, 1)$-INBIBD, a $(52, 9, 6, 2, 5, 1)$-INBIBD and a $(58, 6, 2, 5, 1)$-NGDD of type $7^7 9^1$. For $v = 45$ and 52, fill in the holes with a $(6, 6, 2, 5, 1)$-NBIBD and a $(9, 6, 2, 5, 1)$-NBIBD, which exist by Lemma 3.1. For $v = 58$, fill in the groups with a $(7, 6, 2, 5, 1)$-NBIBD and a $(9, 6, 2, 5, 1)$-NBIBD, which exist by Lemma 3.1. For $v = 69$, by Lemma 3.6, there exists a $(69, 12, 6, 2, 5, 1)$-INBIBD. Then fill in the hole with a $(12, 6, 2, 5, 1)$-NBIBD, which exists by Lemma 3.1. □

Lemma 3.8 *There exists a* $(v, 6, 2, 5, 1)$*-NBIBD for* $v \in \{51, 57, 63, 64, 70, 81, 82, 105, 106, 108, 111, 112, 117, 129, 130, 135, 136, 148, 171, 177, 243, 375, 411\}$*.*

Proof Start from a $(u, 6, 2, 5, 1)$-NBIBD, which is also a $(u, 6, 2, 5, 1)$-NGDD of type 1^u. Apply Construction 2.4 with a TD$(6, m)$ to obtain a $(um, 6, 2, 5, 1)$-NGDD of type m^u. Fill in the groups by Construction 2.1 with an $(m + h, h, 6, 2, 5, 1)$-INBIBD and an $(m + h, 6, 2, 5, 1)$-NBIBD. Then we have an $(mu + h, 6, 2, 5, 1)$-NBIBD. Note that when $h \in \{0, 1\}$, an $(m + h, h, 6, 2, 5, 1)$-INBIBD is just an $(m + h, 6, 2, 5, 1)$-NBIBD. Let $v = mu + h$. Take the parameters (v, u, m, h) as follows, and make use of the above method to obtain all the desired NBIBDs, where the needed TDs are from Lemma 2.2, and the needed NBIBDs are from Lemmas 3.3 and 3.4.

(51, 10, 5, 1),	(57, 7, 8, 1),	(63, 7, 9, 0),	(64, 7, 9, 1),
(70, 10, 7, 0),	(81, 9, 9, 0),	(82, 9, 9, 1),	(105, 15, 7, 0),
(106, 7, 15, 1),	(108, 12, 9, 0),	(111, 10, 11, 1),	(112, 16, 7, 0),
(117, 9, 13, 0),	(129, 16, 8, 1),	(130, 10, 13, 0),	(135, 15, 9, 0),
(136, 15, 9, 1),	(148, 7, 21, 1),	(171, 10, 17, 1),	(177, 16, 11, 1),
(243, 9, 27, 0),	(375, 15, 25, 0),	(411, 10, 41, 1).	

□

Lemma 3.9 *There exists a* $(v, 6, 2, 5, 1)$*-NBIBD for* $v \in \{75, 76, 87, 88, 93, 94, 99, 100, 118, 123, 124, 132, 142, 153, 159, 165, 183, 207, 213, 219, 237, 255, 297, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}$.

Proof Start from a TD$(u+1, m)$. Let $0 \le a \le m$. Give each point weight 1 in the first u groups. In the last group give a points weight x and the remaining $m-a$ points weight 0. If there exist a $(u, 6, 2, 5, 1)$-NGDD of type 1^u (i.e., a $(u, 6, 2, 5, 1)$-NBIBD) and a $(u+x, 6, 2, 5, 1)$-NGDD of type $1^u x^1$ (i.e., a $(u+x, x, 6, 2, 5, 1)$-INBIBD), then applying Construction 2.3, we have an $(mu+ax, 6, 2, 5, 1)$-NGDD of type $m^u(ax)^1$. Fill in the groups by Construction 2.1 with an $(m+h, h, 6, 2, 5, 1)$-INBIBD and a $(ax+h, 6, 2, 5, 1)$-NBIBD. Then we have an $(mu+ax+h, 6, 2, 5, 1)$-NBIBD. Let $v = mu + ax + h$. Take the parameters (v, u, m, a, h, x) as follows, and make use of the above method to obtain all the desired NBIBDs, where the needed TDs are from Lemma 2.2, the needed NBIBDs are from Lemmas 3.3 and 3.4, and the needed $(11, 2, 6, 2, 5, 1)$-INBIBD is from Lemma 3.5.

(75, 6, 11, 8, 1, 1),	(76, 6, 11, 9, 1, 1),	(87, 6, 13, 9, 0, 1),
(88, 6, 13, 10, 0, 1),	(93, 9, 9, 6, 0, 2),	(94, 9, 9, 6, 1, 2),
(99, 9, 9, 9, 0, 2),	(100, 9, 9, 9, 1, 2),	(118, 6, 17, 15, 1, 1),
(123, 6, 19, 9, 0, 1),	(124, 6, 19, 10, 0, 1),	(132, 6, 19, 18, 0, 1),
(142, 12, 11, 9, 1, 1),	(153, 9, 16, 9, 0, 1),	(159, 9, 16, 15, 0, 1),
(165, 9, 17, 11, 1, 1),	(183, 9, 19, 12, 0, 1),	(207, 12, 16, 15, 0, 1),
(213, 12, 17, 8, 1, 1),	(219, 12, 17, 14, 1, 1),	(237, 12, 19, 9, 0, 1),
(255, 6, 41, 8, 1, 1),	(297, 6, 43, 39, 0, 1),	(435, 18, 23, 20, 1, 1),
(453, 12, 37, 9, 0, 1),	(459, 12, 37, 15, 0, 1),	(471, 12, 37, 27, 0, 1),
(489, 9, 53, 11, 1, 1),	(495, 9, 53, 17, 1, 1),	(513, 9, 53, 35, 1, 1),
(519, 9, 53, 41, 1, 1),	(609, 9, 67, 6, 0, 1),	(615, 9, 67, 12, 0, 1),
(621, 9, 67, 18, 0, 1),	(657, 12, 53, 20, 1, 1).	

□

Combining the results of Lemmas 3.3, 3.4, 3.7, 3.8 and 3.9, we complete the proof of Theorem 1.5.

4 Existence of $(v, 6, 3, 5, 2)$-NBIBDs

Lemma 4.1 [9] There is no $(v, 6, 3, 5, 2)$-NBIBD for $v \in \{6, 10\}$. If $v \in \{7, 9, 12, 13, 15, 16\}$, a $(v, 6, 3, 5, 2)$-NBIBD exists.

By Lemma 1.1, a (t, k)GWhD(v) implies a $(v, k, t, k-1, t-1)$-NBIBD. Since when $v \in \{24, 30, 84, 96, 162, 168, 180, 192, 198\}$, a $(3, 6)$GWhD(v) exists [4], we have

Lemma 4.2 *There is a $(v, 6, 3, 5, 2)$-NBIBD for $v \in \{24, 30, 84, 96, 162, 168, 180, 192, 198\}$.*

By Lemma 1.2, if there exists a $(v, k, 1)$-PMD, then there exists a $(v, k, t, k-1, t-1)$-NBIBD for each t satisfying $t \mid k$. Combining the results of Theorem 1.3, Lemmas 4.1 and 4.2, we have

Lemma 4.3 *There is a $(v, 6, 3, 5, 2)$-NBIBD for any integer $v \equiv 0, 1 \pmod 3$ and $v \geq 7$ except for*

(1) $v \in \{18, 48, 54, 60, 72, 90, 102, 108, 114, 132, 138, 150\}$;

(2) $v \equiv 3 \pmod 6$ and $v \in [21, 135] \cup [153, 183] \cup \{207, 213, 219, 237, 243, 255, 297, 375, 411, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}$;

(3) $v \equiv 4 \pmod 6$ and $v \in \{10, 22, 34\} \cup [52, 148]$.

Lemma 4.4 *There is a $(v, 6, 3, 5, 2)$-NBIBD for $v \in \{18, 21, 27, 33\}$.*

Proof For $v \in \{21, 33\}$, let $X = Z_v$ and develop the following base blocks:

$v = 21$: $\{0, 11, 19; 6, 8, 7\}$, $\{0, 3, 8; 15, 4, 19\}$, $\{0, 3, 9; 10, 1, 6\}$, $\{0, 7, 14; 2, 9, 16\}^*$, $(+1 \bmod 21)$;

$v = 33$: $\{0, 6, 12; 13, 22, 30\}$, $\{0, 10, 23; 29, 25, 32\}$, $\{0, 1, 5; 17, 30, 32\}$, $\{0, 1, 15; 20, 23, 28\}$, $\{0, 2, 14; 21, 5, 12\}$, $\{0, 11, 22; 4, 15, 26\}^*$, $(+1 \bmod 33)$.

For $v \in \{18, 27\}$, let $X = Z_{v-1} \cup \{\infty\}$ and develop the following base blocks:

$v = 18$: $\{\infty, 0, 9; 2, 8, 12\}$, $\{0, 1, 15; 2, 4, 7\}$, $\{0, 1, 7; 3, 8, 12\}$, $(+1 \bmod 17)$;

$v = 27$: $\{\infty, 0, 13; 7, 11, 21\}$, $\{0, 1, 8; 2, 22, 23\}$, $\{0, 9, 20; 16, 18, 25\}$, $\{0, 3, 24; 6, 14, 17\}$, $\{0, 4, 16; 13, 17, 3\}^*$, $(+1 \bmod 26)$,

where $\infty + 1 = \infty$. When $v \in \{21, 33\}$, the base blocks with a * will generate $v/3$ distinct blocks under Z_v; when $v = 27$, the base block with a * will generate 13 distinct blocks under Z_{26}. $\square$

Lemma 4.5 *There is a* $(q+t,t,6,3,5,2)$*-INBIBD for* $(q,t)\in\{(9,2),(13,3),(17,4),(61,15)\}$.

Proof By Theorem 1.4, there exists a $(q+t,t,6,1)$-IPMD for $(q,t)\in\{(13,3),(17,4),(61,15)\}$, which implies a $(q+t,t,6,3,5,2)$-INBIBD by Lemma 2.5.

When $(q,t)=(9,2)$, we construct a $(11,2,6,3,5,2)$-INBIBD on $Z_9\cup\{\infty_1,\infty_2\}$ as follows, where $\{\infty_1,\infty_2\}$ is the hole. Develop the two base blocks: $\{0,1,3;4,8,\infty_1\}$ and $\{0,2,5;7,8,\infty_2\}$ by $+1$ modulo 9 to obtain all the blocks, where $\infty_i+1=\infty_i$ for each $i=1,2$. □

Lemma 4.6 *There is a* $(v,6,3,5,2)$*-NBIBD for* $v\in\{52,58,76,150,159,165\}$.

Proof A $(24,3,6,1)$-IPMD, a $(52,9,6,1)$-IPMD, a 6-HPMD of type 3^{10}, a 6-HPMD of type $7^7 9^1$ and a 6-HPMD of type $21^7 9^1$ are constructed explicitly in [1]. Then by Lemma 2.5, there is a $(24,3,6,3,5,2)$-INBIBD, a $(52,9,6,3,5,2)$-INBIBD, a $(30,6,3,5,2)$-NGDD of type 3^{10}, a $(58,6,3,5,2)$-NGDD of type $7^7 9^1$ and a $(156,6,3,5,2)$-NGDD of type $21^7 9^1$.

For $v=52$, fill in the hole of a $(52,9,6,3,5,2)$-INBIBD, where the needed $(9,6,3,5,2)$-NBIBD is from Lemma 4.1. For $v=58$, fill in the groups of a $(58,6,3,5,2)$-NGDD of type $7^7 9^1$, where the needed $(7,6,3,5,2)$-NBIBD and $(9,6,3,5,2)$-NBIBD are from Lemma 4.1. For $v=76$, start from a $(76,15,6,3,5,2)$-INBIBD, which exists by Lemma 4.5. Fill in the hole with a $(15,6,3,5,2)$-NBIBD, which exists by Lemma 4.1. For $v=150$, inflate a $(30,6,3,5,2)$-NGDD of type 3^{10} with a TD$(6,5)$ to obtain a $(150,6,3,5,2)$-NGDD of type 15^{10}. Fill in the groups with a $(15,6,3,5,2)$-NBIBD, which exists by Lemma 4.1. For $v=159$, add three infinite points and fill in the groups of a $(156,6,3,5,2)$-NGDD of type $21^7 9^1$, where the needed $(24,3,6,3,5,2)$-INBIBD exists by the first paragraph of this proof, and the needed $(12,6,3,5,2)$-NBIBD is from Lemma 4.1. For $v=165$, start from a $(11,6,3,5,2)$-NGDD of type $1^9 2^1$, which is also a $(11,2,6,3,5,2)$-INBIBD and exists by Lemma 4.5. Inflate this $(11,6,3,5,2)$-NGDD of type $1^9 2^1$ with a TD$(6,15)$ to obtain a $(165,6,3,5,2)$-NGDD of type $15^9 30^1$. Fill in the groups with a $(15,6,3,5,2)$-NBIBD and a $(30,6,3,5,2)$-NBIBD, which exist by Lemmas 4.1 and 4.2. □

Lemma 4.7 *There is a* $(v,6,3,5,2)$*-NBIBD for* $v\in\{57,63,81,94,100,105,106,108,112,117,123,129,135,136,153,171,177,243,375,513,657\}$.

Proof Start from a $(u,6,3,5,2)$-NBIBD, which is also a $(u,6,3,5,2)$-NGDD of type 1^u. Apply Construction 2.4 with a TD$(6,m)$ to obtain a $(um,6,3,5,2)$-NGDD of type m^u. Fill in the groups by Construction 2.1 with an $(m+h,h,6,3,5,2)$-INBIBD and an $(m+h,6,3,5,2)$-NBIBD. Then we have an

$(mu+h, 6, 3, 5, 2)$-NBIBD. Note that when $h \in \{0, 1\}$, an $(m+h, h, 6, 3, 5, 2)$-INBIBD is just an $(m+h, 6, 3, 5, 2)$-NBIBD. Let $v = mu+h$. Take the parameters (v, u, m, h) as follows, and make use of the above method to obtain all the desired NBIBDs, where the needed TDs are from Lemma 2.2, the needed NBIBDs are from Lemmas 4.3 and 4.4, the needed $(16, 3, 6, 3, 5, 2)$-INBIBD and $(21, 4, 6, 3, 5, 2)$-INBIBD are from Lemma 4.5.

(57, 7, 8, 1),	(63, 7, 9, 0),	(81, 9, 9, 0),	(94, 7, 13, 3),
(100, 9, 11, 1),	(105, 15, 7, 0),	(106, 7, 15, 1),	(108, 12, 9, 0),
(112, 16, 7, 0),	(117, 9, 13, 0),	(123, 7, 17, 4),	(129, 16, 8, 1),
(135, 15, 9, 0),	(136, 9, 15, 1),	(153, 19, 8, 1),	(171, 9, 19, 0),
(177, 16, 11, 1),	(243, 9, 27, 0),	(375, 15, 25, 0),	(513, 19, 27, 0),
(657, 9, 73, 0).			

□

Lemma 4.8 *There is a $(v, 6, 3, 5, 2)$-NBIBD for $v \in \{93, 99, 114, 118, 124, 130, 132, 138, 142, 183, 207, 213, 219, 237, 255, 297, 411, 435, 453, 459, 471, 489, 495, 519, 609, 615, 621\}$.*

Proof Start from a TD$(u+1, m)$. Let $0 \le a \le m$. Give each point weight 1 in the first u groups. In the last group give a points weight x and the remaining $m-a$ points weight 0. If there exist a $(u, 6, 3, 5, 2)$-NGDD of type 1^u (i.e., a $(u, 6, 3, 5, 2)$-NBIBD) and a $(u+x, 6, 3, 5, 2)$-NGDD of type $1^u x^1$ (i.e., a $(u+x, x, 6, 3, 5, 2)$-INBIBD), then applying Construction 2.3, we have an $(mu+ax, 6, 3, 5, 2)$-NGDD of type $m^u(ax)^1$. Fill in the groups by Construction 2.1 with an $(m+h, h, 6, 3, 5, 2)$-INBIBD and a $(ax+h, 6, 3, 5, 2)$-NBIBD. Then we have an $(mu+ax+h, 6, 3, 5, 2)$-NBIBD. Let $v = mu+ax+h$. Take the parameters (v, u, m, a, h, x) as follows, and make use of the above method to obtain all the desired NBIBDs in this lemma, where the needed TDs are from Lemma 2.2, the needed NBIBDs are from Lemmas 4.3 and 4.4, and the needed $(11, 2, 6, 3, 5, 2)$-INBIBD and $(16, 3, 6, 3, 5, 2)$-INBIBD are from Lemma 4.5.

(93, 9, 9, 6, 0, 2),	(99, 9, 9, 9, 0, 2),	(114, 9, 11, 7, 1, 2),
(118, 9, 11, 9, 1, 2),	(124, 9, 13, 2, 3, 2),	(130, 9, 13, 5, 3, 2),
(132, 9, 13, 6, 3, 2),	(138, 9, 13, 9, 3, 2),	(142, 9, 13, 11, 3, 2),
(183, 9, 19, 6, 0, 2),	(207, 9, 19, 18, 0, 2),	(213, 12, 17, 8, 1, 1),
(219, 12, 17, 14, 1, 1),	(237, 12, 19, 9, 0, 1),	(255, 15, 16, 15, 0, 1),
(297, 15, 19, 12, 0, 1),	(411, 9, 43, 12, 0, 2),	(435, 18, 23, 20, 1, 1),
(453, 12, 37, 9, 0, 1),	(459, 12, 37, 15, 0, 1),	(471, 12, 37, 27, 0, 1),
(489, 13, 36, 7, 0, 3),	(495, 13, 36, 9, 0, 3),	(519, 9, 49, 39, 0, 2),
(609, 12, 49, 21, 0, 1),	(615, 12, 49, 27, 0, 1),	(621, 9, 67, 9, 0, 2).

□

Combine the results of Lemmas 4.3, 4.4, 4.6, 4.7 and 4.8. Then we have Theorem 1.6.

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Received: November 13, 2011