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WATER WAVE SCATTERING BY AN UNEVEN DOCK

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Abstract

Water wave scattering by an uneven dock is investigated here. Using a simplified perturbation analysis, the first order correction to the reflection coefficient is obtained in terms of an integral involving the shape function describing the dock topography. The first order reflection coefficient is depicted graphically for two shape functions of the dock against the wave number.

Mathematics Subject Classification: 76B.

Keywords: Uneven dock, Water wave scattering, Reflection coefficient.

1 Introduction

Dock problems are concerned with the interaction of water waves with a thin rigid plate which may be of infinite or finite length lying on the free surface of ocean. These problems are interesting due to the reason that the simple geometry of the dock problem allows a considerable progress in developing mathematical as well as numerical techniques. If the dock is of semi-infinite extent, then the problem yields explicit solution. Problem of scattering of obliquely incident waves by semi-infinite dock was investigated by Heins [1] by employing a method based on Wiener-Hopf technique. Friedrichs and Lewy [2] used complex variable technique to solve the problem of scattering of normally incident waves on a semi-infinite dock. Chakrabarti et al. [3] reexamined this problem by utilizing a Fourier type of analysis, giving rise to Carleman type singular integral equations over semi-infinite intervals and recovered known results. Problems related to finite dock are investigated by a number of researchers [4], [5], [6], [7] by using various mathematical techniques for normal as well as oblique incidence cases.

This paper is concerned with the scattering of surface water waves by a semi-infinite uneven plate floating on the free surface of infinitely deep water which give rise to a mixed boundary value problem, involving Laplace equation in two dimension. To solve the corresponding boundary value problem, a perturbation method is employed directly to the governing partial differential equation, the free surface condition, the boundary condition, the bottom condition and the infinity conditions satisfied by the potential function describing the fluid motion. This procedure produces a series boundary value problems (BVPs) for potential functions of increasing orders, of which we consider only the first two BVPs, viz. the zero-order and the first-order. The BVP for the zero-order potential function is concerned with the problem of scattering of water waves by a flat rigid dock. This is a classical problem in the linearised theory of water waves and was investigated by [2] and reexamined by [3]. The BVP for the first-order potential function satisfies a radiation problem involving a semi-infinite dock. The quantity of physical interest in this BVP, namely the first order reflection coefficient, is obtained by using the Green's integral theorem, in terms of an integral involving the shape function describing the dock topography and the solution of zero-order BVP. For obtaining numerical results the shape of the dock is chosen in the form of a sinusoidal curve of finite extent followed by a semi-infinite straight line and of a Gauss type curve. The first order reflection coefficient for these shapes is calculated numerically and depicted graphically against the wave number in a number of figures. It is observed that for a sinusoidal form of the dock topography, the reflection of wave energy up to first-order increases as the number of ripples in the dock topography increases.

2 Formulation of the Problem

It is assumed that water is an incompressible, inviscid and homogeneous fluid and that the motion in the fluid starts from rest so that it is irrotational and can be described by a velocity potential ϕ. We consider water of infinite depth and y-axis being taken vertically downwards into the fluid. The semi-infinite uneven dock occupies the position $y = \epsilon\ c(x)$, $x \geq 0$. Here, $c(x)$ is a continuous and a bounded function of x and vanishes at $x = 0$ and as $x \to \infty$ and ϵ is a dimensionless small parameter giving the measure of maximum deviation (small) of the uneven dock from its mean horizontal position. A progressive wave train propagating from the direction of $x = -\infty$ and described by the complex velocity potential function $e^{-Ky+iKx}$ is incident on the dock. It is fully reflected by the dock. Assuming linear theory and irrotational motion, the velocity potential describing the fluid motion can be represented as $\mathrm{Re}\{(g^2/\sigma^3)\phi(x,y)e^{-i\sigma t}\}$ where σ is the circular frequency and g is the acceleration due to gravity. Then, $\phi(x,y)$ satisfies

$$\nabla^2\phi = 0 \qquad \text{in the fluid region,} \tag{2.1}$$

the free surface condition

$$K\phi + \phi_y = 0 \qquad \text{on } \ y = 0, \ \ x < 0, \tag{2.2}$$

where

$$K = \sigma^2/g$$

the dock condition

$$\phi_n = 0 \qquad \text{on } \ y = \epsilon c(x), \tag{2.3}$$

n denoting the normal derivative,
the bottom condition

$$\nabla\phi \to 0 \qquad as \ \ y \to \infty, \tag{2.4}$$

the edge condition

$$r\frac{\partial\phi}{\partial r} = 0 \qquad asr = (x^2+y^2)^{1/2} \to 0, \tag{2.5}$$

and the infinity conditions

$$\phi(x,y) = \begin{cases} e^{-Ky+iKx} + Re^{-Ky-iKx} & \text{as } \ x \to -\infty, \\ 0 & \text{as } \ x \to \infty. \end{cases} \tag{2.6}$$

where R is the reflection coefficient.
The dock condition (2.3) can be expressed approximately as

$$\phi_y - \epsilon c'(x)\phi_x + O(\epsilon^2) \quad \text{on } \ y = 0, x > 0 \tag{2.7}$$

This suggests that a perturbation expansion for the velocity potential $\phi(x,y)$ can be employed to solve the BVP described by (2.1) to (2.6) approximately. This is described in the next section.

3 Method of Solution

The approximate boundary condition (2.7) suggests that ϕ and R can be expanded in terms of ϵ as given by

$$\phi(x,y;\epsilon) = \phi_0(x,y) + \epsilon\phi_1(x,y) + O(\epsilon^2),$$

$$R(\epsilon) = R_0 + \epsilon R_1 + O(\epsilon^2). \tag{3.1}$$

Substituting the expansions (3.1) in (2.1),(2.2),(2.4)-(2.6) and (2.7) we find after equating the coefficients of ϵ^0 and ϵ^1 from both sides, that the functions $\phi_0(x,y)$ and $\phi_1(x,y)$ satisfy the following BVPs:
BVP-I: The function $\phi_0(x,y)$ satisfies

$$\nabla^2\phi_0(x,y) = 0 \quad \text{in the fluid region } y > 0,$$

$$K\phi_0(x,y) + \phi_{0y}(x,y) = 0 \quad \text{on} \quad y = 0, x < 0,$$

$$\phi_{0y}(x,y) = 0 \quad \text{on} \quad y = 0, x > 0,$$

$$r\frac{\partial\phi_0}{\partial r} = 0 \quad \text{as} \quad r = (x^2+y^2)^{1/2} \to 0,$$

$$\nabla\phi_0(x,y) \to 0 \quad \text{as} \quad y \to \infty,$$

$$\phi_0(x,y) = \begin{cases} e^{-Ky+iKx} + R_0 e^{-Ky-iKx} & as \quad x \to -\infty, \\ 0 & as \quad x \to \infty. \end{cases} \tag{3.2}$$

BVP-II: The function $\phi_1(x,y)$ satisfies

$$\nabla^2\phi_1(x,y) = 0 \quad \text{in the fluid region } y > 0,$$

$$K\phi_1(x,y) + \phi_{1y}(x,y) = 0 \quad \text{on} \quad y = 0, x < 0,$$

$$\phi_{1y} = \frac{d}{dx}(c(x)\frac{\partial\phi_0(x,0)}{\partial x}) \quad \text{on} \quad y = 0, x > 0,$$

$$r\frac{\partial\phi_1}{\partial r} = 0 \quad \text{as} \quad r = (x^2+y^2)^{1/2} \to 0,$$

$$\nabla\phi_1(x,y) \to 0 \quad \text{as} \quad y \to \infty,$$

$$\phi_1(x,y) = \begin{cases} R_1 e^{-Ky-iKx} & \text{as} \quad x \to -\infty, \\ 0 & \text{as} \quad x \to \infty. \end{cases} \tag{3.3}$$

It may be noted that the BVP-I corresponds to the classical problem of water wave scattering by a semi-infinite plane dock. This was solved earlier by [2] reexamined by [3]. The BVP-II is a radiation problem involving a dock. Without solving $\phi_1(x,y)$ explicitly, R_1 can be determined in terms of an integral involving the shape function $c(x)$ and $\phi_{0x}(x,0)$. To show this, we apply Green's integral theorem to the functions $\phi_0(x,y)$ and $\phi_1(x,y)$ in the region bounded by the lines $y = 0, (0 \le x \le X)$; $x = X, (0 \le y \le Y)$; $y = Y, (-X \le x \le X)$; $x = -X, (0 \le y \le Y)$ and $y = 0, (-X \le x \le 0)$. Denoting by l the contour which bounds the region we have,

$$\int_l (\phi_0\frac{\partial\phi_1}{\partial n} - \phi_1\frac{\partial\phi_0}{\partial n})dl = 0 \tag{3.4}$$

where n denotes the outward drawn normal to the line element dl. The free surface condition and the bottom condition satisfied by ϕ_0 and ϕ_1 ensure that there is no contribution to the integral on the left side of the equation (3.4) from the free surface and from the line $y = Y, (-X \leq x \leq X)$ as $Y \to \infty$. As both ϕ_0 and $\phi_1 \to 0$ *as* $X \to \infty$ there is no contribution from the line $x = X, (0 \leq y \leq Y)$ as $X \to \infty$. The only contributions arise from the line integral along the line $y = 0, (0 \leq x \leq X)$ and from the incoming wave part in $\phi_0(x, y)$ combining with the outgoing wave part in $\phi_1(x, y)$ for the line $x = -X, (0 \leq y \leq Y)$. Thus, we find that

$$R_1 = i \int_0^\infty c(x)\phi_{0x}^2(x, 0)dx \tag{3.5}$$

Thus, R_1 is derived in terms of integral involving the shape function $c(x)$ and the zero-order potential function. It is known that (cf.[3])

$$\phi_0(x, y) = \frac{2}{\pi}\int_0^\infty \frac{B(\xi)}{\xi} \cos\xi y \; e^{-\xi x} d\xi \qquad for \quad x > 0 \tag{3.6}$$

and

$$R_0 = exp\Big(\frac{-i\pi}{4}\Big)$$

where

$$B(\xi) = \frac{K}{2}\;\Phi_0(-\xi)\Big(\frac{1}{(\xi + iK)\;\Phi_0(iK)} + \frac{R}{(\xi - iK)\;\Phi_0(-iK)}\Big),$$

$$\Phi_0(\zeta) = \exp\left(\frac{1}{2\pi i}\int_0^\infty \frac{\ln(\frac{u+iK}{u-iK})}{u - \zeta}\,du\right) \qquad (\zeta = \xi + i\eta).$$

$B(\xi)$ can be expressed as

$$B(\xi) = \frac{K\xi}{\xi^2 + K^2}\;\exp\left(\frac{-(\xi + iK)}{2\pi i}\int_0^\infty \frac{\ln(\frac{u+iK}{u-iK})}{(u + \xi)(u - iK)}\,du\right).$$

Thus, we obtain

$$\begin{aligned}\phi_{0x}(x, 0) &= \frac{-2K}{\pi}\int_0^\infty \frac{\xi}{\xi^2 + K^2}\;\exp\left(\frac{-(\xi + iK)}{2\pi i}\int_0^\infty \frac{\ln(\frac{u+iK}{u-iK})}{(u + \xi)(u - iK)}\,du\right)e^{-\xi x}d\xi \\ &= \frac{-2K}{\pi}\int_0^\infty \frac{\sqrt{\xi}\;e^{-\xi x}}{\xi^2 + K^2}\;\exp\left(\frac{-i\pi}{8}\right)\;\exp\left(\frac{1}{\pi}\int_0^{\pi/2} log(K\sin\theta + \xi\cos\theta)d\theta\right)d\xi.\end{aligned} \tag{3.7}$$

Substituting this in (3.5) we obtain R_1 as

$$R_1 = \frac{2\sqrt{2}\;(1 + i)\;K^2}{\pi^2}\int_0^\infty c(x)\Big(\int_0^\infty \frac{\sqrt{\xi}\;e^{-\xi x}}{\xi^2 + K^2}\;\exp\left(\frac{1}{\pi}\int_0^{\pi/2} log(K\sin\theta + \xi\cos\theta)d\theta\right)d\xi\Big)^2 dx. \tag{3.8}$$

R_1 in (3.8) can now be computed numerically once the shape function $c(x)$ is known. Here, first we consider a dock with sinusoidal variation for which $c(x)$ is chosen in the form

$$c(x) = \begin{cases} a\sin\lambda x, & 0 \le x \le \frac{n\pi}{\lambda}, \\ 0 & otherwise. \end{cases} \tag{3.9}$$

where n is a positive integer. Thus, there exists n number of sinusoidal ripples at the dock with wave number λ. Next we choose $c(x)$ in the form

$$c(x) = xe^{-\mu^2\ x^2}, \qquad x > 0.$$

4 Numerical Results

In this section we have computed $|R_1|$ for different values of wave number Kh where h is the characteristic length used to non- dimensionalise K, for two types of shape functions $c(x)$ characterizing the unevenness of the dock as mentioned earlier. For numerical computation the value of the non-dimensional parameter $\frac{a}{h}$ is taken as 1.0 in figs.1 and 2 which depicts $|R_1|$ against Kh for $c(x)$ given by (3.9).

In fig.1, $|R_1|$ is plotted against Kh when the shape of the dock is sinusoidal of wavelength $\frac{2\pi}{\lambda}$, having a number of ripples n=2,5,7 and 10 and $\lambda h = 1.0$. It is observed from fig. 1 that for $0 < Kh < 2$, $|R_1|$ increases sharply and reaches a peak and then decreases as Kh further increases. It is also observed that the peak value in $|R_1|$ becomes higher as n increases from 2 to 10. This shows that for a fixed wavelength of the ripples in the dock, $|R_1|$ increases as the number of ripples increases. In fig.2, the value of the non-dimensional parameter λh is taken as 2.0. Similar nature of $|R_1|$ is observed as seen in fig. 1 except that the peak values of $|R_1|$ is slightly higher. Thus for sinusoidal form of dock, reflection of wave energy upto first order increases as the number of ripples in the dock is increased.

In fig.3, $|R_1|$ is plotted against Kh for $c(x)$ given by (3.10) for the values of the non-dimensional parameter μh as 0.1,0.2 and 0.3. It is observed that for $0 < Kh < 0.5$ $|R_1|$ increases and then decreases to become almost constant as Kh further increases. It is also observed that for fixed values of Kh, $|R_1|$ decreases as μh increases from 0.1 to 0.3. This shows that increase of μh causes decrease in $|R_1|$.

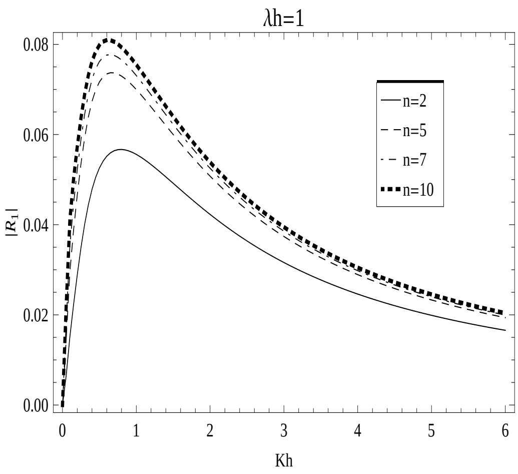

Fig.1:First order reflection coefficient

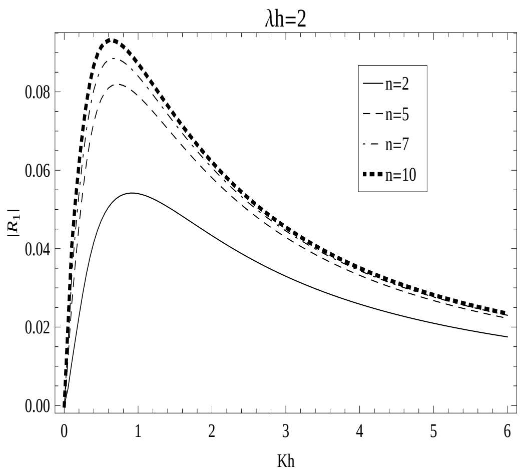

Fig.2:First order reflection coefficient

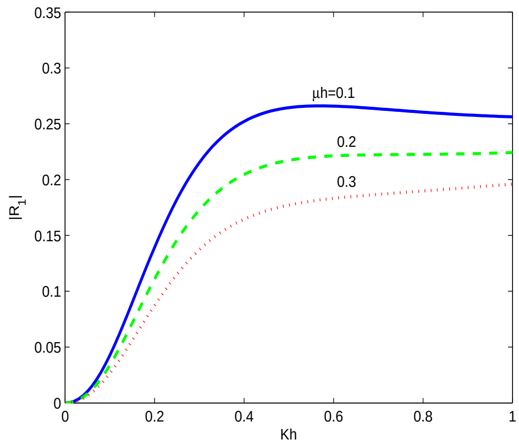

Fig.3:First order reflection coefficient

5 Conclusion

A simplified perturbation analysis together with appropriate use of Green's integral theorem is employed to obtain the first order correction to the reflection coefficient for the problem of water wave scattering by an uneven dock. The dock topography is described by sinusoidal curve of finite extent and also by a Gauss type curve. And for both the profiles the first order reflection coefficient is depicted graphically against the wave number.

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References

[1] A. E. Heins, Water waves over a channel of finite depth with a dock, American Journal of Mathematics, 70 (1948), 730-748.

[2] K.O. Friedrichs and H. Lewy, The Dock Problem, Commun. Pure Appl. Math., 1 (1948), 135-148.

[3] A. Chakrabarti, B.N. Mandal and Rupanwita Gayen, The Dock Problem Revisited, International Journal of Mathematics and Mathematical Sciences, 21 (2005), 3459-3470.

[4] R. L. Holford, Short surface waves in the presence of a finite dock, I. Proc. Camb. Phil. Soc., 60 (1964), 957-983.

[5] R. L. Holford, Short surface waves in the presence of a finite dock, II. Proc. Camb. Phil. Soc. 60 (1964), 985-1011.

[6] H. Rubin, The dock of finite extent, Commun. Pure Appl. Math., 7 (1954), 317-344.

[7] C. M. Linton, The finite dock problem, J. Applied Math. Phys., (ZAMP). 52 (2001), 640-656.

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